Symmetric Functions + Schubert Polys

Symmetric Polynomials

$R = \mathbb{Z}[x_1, \dots, x_n]$

Definition $p \in R$ is symmetric if $p(x_1, \dots, x_n) = p(x_{\pi(1)}, \dots x_{\pi(n)})$ $\forall \pi \in S_n$.

Examples ($n=2$) ① $x_1 + x_2$ ✓

② x_1 ×

③ any constant ✓

(general n) ④ $\sum_{i=1}^{n} x_i$ ✓

⑤ $\sum_{i=1}^{n} x_i^{18}$ ✓

⑥ $\sum_{i \le j} x_i x_j$ ✓

⑦ $\sum_{i<j} x_i x_j$ ✓

Question: What are generators for the subring R^{S_n} of symmetric polys?

Even generating $\mathbb{Z}[x_1, x_2]$, $\exists$ many choices. So we will insist that the generators are homogeneous

Definition: A polynomial is homogeneous if all terms have the same degree.

(This wouldn't be possible if we looked at $R' = \mathbb{Z}[x_1 + x_2, x_1 + x_2^2]$

One guess for $n=2$ would be $x_1 + x_2$ and $x_1^2 + x_2^2$.

This almost works: if $p \in \mathbb{Z}[x_1, x_2]^{S_2}$, then $\exists q \in \mathbb{Q}[a, b]$ s.t.

$p(x_1, x_2) = q(x_1 + x_2, x_1^2 + x_2^2)$

Definition $e_k := \sum_{i_1 < i_2 < \dots < i_k} x_{i_1} x_{i_2} \dots x_{i_k}$, $(1 \le i_1 < \dots < i_k \le n)$ $(k \le n)$

is the k^{th} elementary symmetric poly

Theorem $\forall p \in R^{S_n}$ $\exists!\, q \in \mathbb{Z}[a_1, \ldots, a_n]$ s.t. $q(e_1, \ldots, e_n) = p$.

Example $n = 2$, $(e_1 = x_1 + x_2,\ e_2 = x_1 x_2)$
if $p = x_1 + x_2$, $q(a, b) = a$ is the corresponding poly
N.B not symmetric

Definition given p, the *initial term*, $\operatorname{init}(p)$, is the
(*) term of largest degree, with most x_1s
(if $\exists$ a tie) with most x_2s ...etc

Lemma: If p is symmetric, ie if $p \in R^{S_n}$, then $p = \sum_{k=1}^{\deg(p)} p_k$ where p_k is homogeneous of degree k and symmetric.

Proof: let $d = \deg(p)$, ~~and~~

$$\therefore \quad p_d = \lim_{t \to \infty} \frac{p(tx_1, \ldots, tx_n)}{t^d}$$

So p_d is symmetric. Hence $p - p_d$ is symmetric and $p - p_d = \sum_{k=0}^{d-1} p_k$, symmetric of lower degree $\to$ result follows by induction.

Proof of Theorem: By the Lemma, we can assume that p is homogeneous.
let $\operatorname{init}(p) = c \prod_{i=1}^{n} x_i^{m_i}$. Since p is symmetric, $m_1 \geq m_2 \geq \ldots \geq m_n$.
Define $m_{n+1} = 0$.

Replace p by $p - c \prod_{i=1}^{n} e_i^{m_i - m_{i+1}}$

Note that $\operatorname{init}(p) = \operatorname{init}\left(c \prod_{i=1}^{n} e^{m_i - m_{i+1}}\right)$

So $p - c \prod_{i=1}^{n} e^{m_i - m_{i+1}}$ is again symmetric and homogeneous, but "later in the dictionary" (wrt ordering above (*)).
By induction on $\operatorname{init}(p)$, $p = q(e_1, \ldots, e_n)$ for some unique q. $\blacksquare$

2 Schubert Polynomials

How to measure, given $p \in R$, how non-symmetric p is in x_i, x_{i+1} ?

Answer 1: $p(x_1, \dots, x_i, x_{i+1}, \dots, x_n) - p(x_1, \dots, x_{i+1}, x_i, \dots, x_n)$

$$=: p - r_i p$$

for operators $r_1, \dots, r_{n-1} : R \longrightarrow R$.
(ring automorphisms)

note $r_i r_j = r_j r_i$ if $|i-j| > 1$.

Note $(p - r_i p)\big|_{x_i = x_{i+1}} = 0$

$\therefore \; x_i - x_{i+1} \mid p - r_i p$

Def^n: $\partial_i p = \dfrac{p - r_i p}{x_i - x_{i+1}}$

Again a polynomial with integer-coefficients!

Call ∂_i the "divided difference operation"

Easy properties: ① ∂_i has degree -1

② $\partial_i^2 = 0$

③ if $\partial_i p = 0$, then $\partial_i (pq) = p\, \partial_i q$

④ $\partial_i \partial_j = \partial_j \partial_i$ if $|i-j| > 1$

⑤ $\partial_i \partial_{i+1} \partial_i = \partial_{i+1} \partial_i \partial_{i+1}$

Examples:

$\partial_2 x_1 = 0$

$$\begin{tikzcd}
 & & & x_1^2 x_2 \arrow[dll, "\partial_1"'] \arrow[drr, "\delta_2"] & & & \\
 & x_1 x_2 = \frac{x_1^2 x_2 - x_2^2 x_1}{x_1 - x_2} \arrow[dl, "\partial_1"'] \arrow[dr, "\partial_2"] & & & & x_1^2 \arrow[dl, "\partial_1"'] \arrow[dr, "\delta_2"] & \\
0 & & x_1 \arrow[dr, "\partial_1"] & & x_1 + x_2 = \frac{x_1^2 - x_2^2}{x_1 - x_2} \arrow[dl, "\partial_2"'] & & 0 \\
 & & & 1 & & &
\end{tikzcd}$$