Polynomials

26/7/12

Example: 1432 → 4132 → 4312 → 4321, so

$$\mathfrak{S}_{1432} = \partial_1\partial_2\partial_3\,\mathfrak{S}_{4321}$$
$$= \partial_1\partial_2\partial_3\, x_1^3x_2^2x_3$$
$$= \partial_1\partial_2\, x_1^3x_2^2$$
$$= \partial_1\left(x_1^3(x_2+x_3)\right)$$
$$= \partial_1(x_1^3x_2) + \partial_1(x_1^3x_3)$$
$$= x_1x_2\,\partial_1(x_1^2) + x_3\,\partial_1(x_1^3)$$
$$= x_1x_2(x_1+x_2) + x_3(x_1^2+x_1x_2+x_2^2)$$

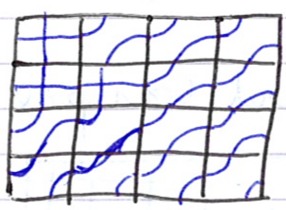
all positive coefficients !!!

Where we are heading:
1. A proof that these are positive
2. A (positive) formula for what they count.

Def^n: tiles ⊞ a cross and an elbow.

Def^n: A pipe dream is an $n\times n$ grid of tiles s.t.
1. in (lower right half) including diagonal are only elbows.
2. no two pipes cross twice.

For example:



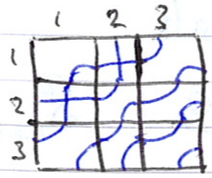
To a pipe dream can associate:
1. A monomial $\prod_{i=1}^{n} x_i^{\#\boxplus \text{ in row } i}$
2. reduced word

start (∴ a perm)

end

Each + gives a number, depending on its antidiagonal

Non-example:

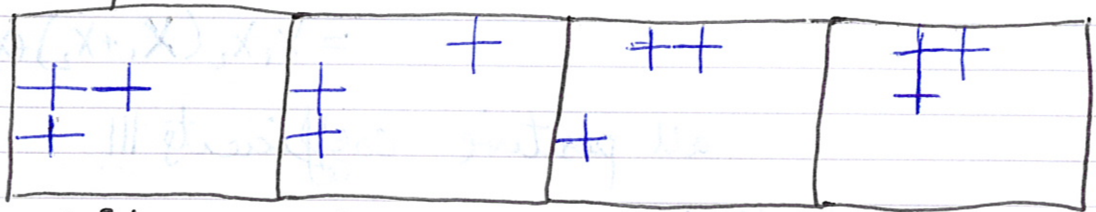


gives 22

Thm to come

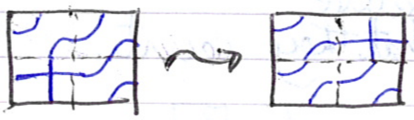
$$\mathfrak{S}_\pi = \sum_{\substack{\text{pipe} \\ \text{dreams } P \\ \text{for that perm}}} \prod_{i=1}^{n} x_i^{\#\text{crosses in row } i}$$

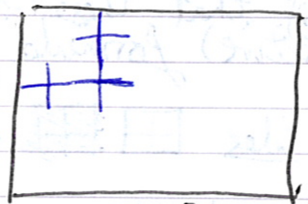
Example: 1432



$x_2^2 x_3$ $\quad$ $x_1 x_2 x_3$ $\quad$ $x_1^2 x_3$ $\quad$ $x_1^2 x_3$

moves:

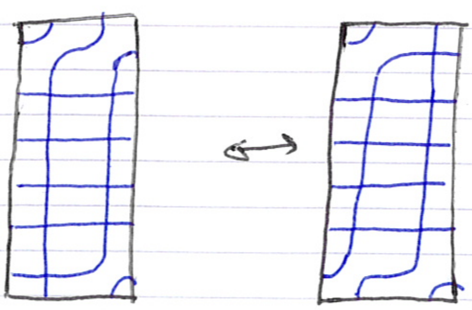




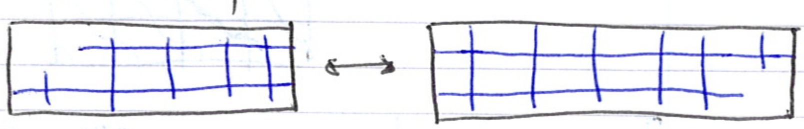
$x_1 x_2^2$

[What turns out to be enough moves:

[Bergeron-Billey '93]



and transpose:


]

Thm: The product of two Schubert polys, expanded in the basis of Schubert polys, has positive coefficients.

Proof: Algebraic topology and algebraic geometry QED.

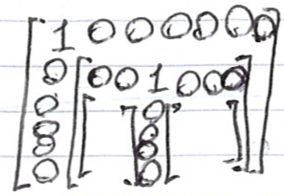
Question: What do the coefficients count?

Unsolved

What's a natural origin of pipe dreams?

Side Topic:

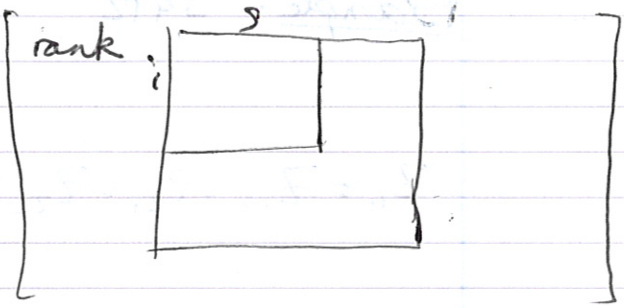
Question: How nice can we make a matrix look, using downward row operations and rightward column operations?



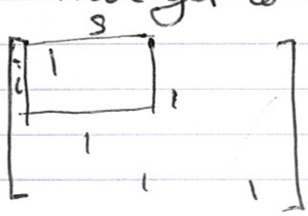
Answer: Can always get a "partial perm matrix", $\leq$ one 1 in each row & column.

Question: Given M, how do we determine which perm matrix we'll get?

Answer: Look at all these numbers: rank



When we get to



that rank = # 1s in the reduced form.

Example: $\pi = \underline{2143}$

$$\begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

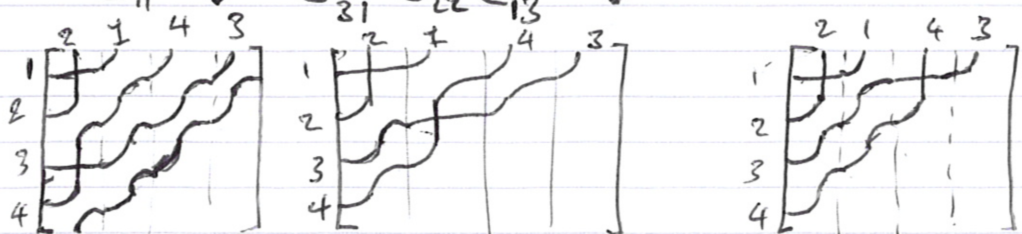
Which matrices are related to this by ~~col~~ downward row and rightward column ops?

$\cancel{z_{11}=0}$ and $\det \begin{vmatrix} z_{11} & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & z_{33} \end{vmatrix} = 0$

$z_{11} = 0$

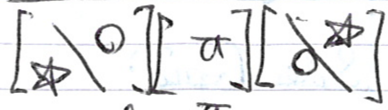
Crazy idea: replace all dets by their antidiagonal terms

$z_{11} = 0 \quad z_{31} z_{22} z_{13} = 0$



$z_{11} = z_{31} = 0 \qquad z_{11} = z_{22} = 0 \qquad z_{11} = z_{13} = 0$

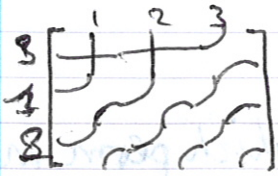
$x_1 x_3 + x_1 x_2 + x_1^2 = \mathfrak{S}_{2143}$

Thm: [K - Miller 2005]
If you replace the dets describing  by their anti-diagonal terms, the components of the results $\longleftrightarrow$ pipe dreams for π.

Example: $\pi = 312$

$$\begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

$$z_{11} = z_{12} = 0$$



rank $\square$ $\leq r$

$\Leftrightarrow$ every $(r+1) \times (r+1)$ det vanishes

Example: 3412

$$\begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

$$z_{11} = z_{12} = z_{21} = z_{22} = 0$$