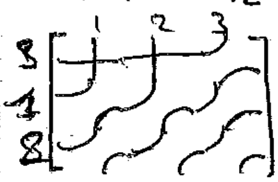
Thm: [K - Miller 2005]

If you replace the dets describing $\left[\begin{smallmatrix} * & \diagdown 0 \end{smallmatrix}\right]\left[\pi\right]\left[\begin{smallmatrix} \diagdown & * \end{smallmatrix}\right]$ by their anti-diagonal terms, the components of the results $\longleftrightarrow$ pipe dreams for π.

Example: $\pi = 312$

$$\begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

$z_{11} = z_{12} = 0$



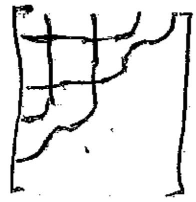
rank $\boxed{} \le r$

$\Leftrightarrow$ every $(r+1)\times(r+1)$ det vanishes

Example: 3412

$$\begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

$z_{11} = z_{12} = z_{21} = z_{22} = 0$



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$p \overset{\text{want}}{=} \sum c_\pi \mathfrak{S}_\pi$, where p is already homogeneous.

How to compute c_π?

$$c_\pi = \partial_\pi p$$

Magic fact: $\mathfrak{S}_\pi \mathfrak{S}_\rho = \sum \underbrace{c^\sigma_{\pi\rho}}_{\geq 0} \mathfrak{S}_\sigma$

$$c^\sigma_{\pi\rho} = \partial_\sigma(\mathfrak{S}_\pi \mathfrak{S}_\rho)$$

Monk's Rule (1950)

$$\mathfrak{S}_\pi \mathfrak{S}_{(i \leftrightarrow i+1)} = \sum 1 \cdot \mathfrak{S}_{\pi'}$$

$\pi' = \pi \circ (a \leftrightarrow b)$
switch positions a & b.
$a \le i < i+1 \le b$, $\pi(a) < \pi(b)$
$\nexists\, c$, $a \le c \le b$, $\pi(a) < \pi(c) < \pi(b)$

Aside: This ~~co~~ came up before Schubert poly's! As Monk was working on $\mathbb{Z}[x_1, \ldots, x_n]/\langle$symmetric poly's of +ve degree$\rangle$.

$\underline{\text{Ex.}}$ $\mathfrak{S}_{\underline{3621}|\underline{54}789\ldots}\ \mathfrak{S}_{(4\leftrightarrow 5)} = \mathfrak{S}_{\underline{562134}} + \mathfrak{S}_{\underline{462153}} + \mathfrak{S}_{\underline{3721546}}$

$+ \mathfrak{S}_{\underline{365124}} + \mathfrak{S}_{\underline{364152}} + \mathfrak{S}_{\underline{362514}}$

$+ \mathfrak{S}_{\underline{362451}}$ /.

$\mathfrak{S}_{(4\leftrightarrow 5)}$

$\underbrace{\underbrace{1\,2\,3\,\cancel{5}}_{\text{symm}}\underbrace{4\,6\,7\,8\ldots}_{\text{symm.}}}$ $\therefore$ doesn't use those

$= \underset{\text{const.}}{c}\,(x_1 + x_2 + x_3 + x_4)$

$\underline{\text{Lemma}}$: $\mathfrak{S}_{(i\leftrightarrow i+1)} = x_1 + \cdots + x_i$.

$\underline{\text{Proof (Sketch)}}$: Both sides are homogeneous of same degree $\ell(\pi)+1 > 0$.

Enough to show $=$ after applying any ∂_i.

$\underline{\text{Recall}}$: $\partial_i p = 0 \Rightarrow \partial_i(pq) = p\,\partial_i(q)$. "$\square$"

$\underline{\text{Corollary}}$: $\underbrace{x_i}_{(\mathfrak{S}_{(i+1\leftrightarrow i)} - \mathfrak{S}_{(i-1\leftrightarrow i)})} \mathfrak{S}_\pi = \sum \mathfrak{S}_{\pi'} - \sum \mathfrak{S}_{\pi'}$

$\pi' = \pi\circ(i\leftrightarrow b)$ same rule $i < b$; $\pi' = \pi\circ(i\leftrightarrow a)$ same rule $i > a$.

$\underline{\text{Lascoux's transition formula}}$ ('76?)

Let $\pi \neq Id$, d = last descent of π. Let $e > d$ be the ~~first~~ last place s.t. ~~$\pi(e) > \pi(d)$~~ $\pi(e) < \pi(d)$

Let $\rho = \pi\circ(d \leftrightarrow e)$ $\therefore \ell(\rho) = \ell(\pi) - 1$

Then $\mathfrak{S}_\pi = x_d\,\mathfrak{S}_\rho + \underbrace{\sum_{\rho'} \mathfrak{S}_{\rho'}}_{\text{Monk's Rule Terms.}}$

$\underline{\text{Pf}}$: d, e were carefully chosen so that in Cor to Monk's, $\exists!$ positive term, $\mathfrak{S}_\pi$. $\square$

$\underline{\text{Cor}}$: ~~Sur~~ Schubert poly's have ≥ 0 coefficients

$\underline{\text{Proof}}$: By induction (in some way.) $\square$

One can use this to prove the pipe dream formula.

Def^n: init p = monomial of highest degree

highest power of x_3, and if tie

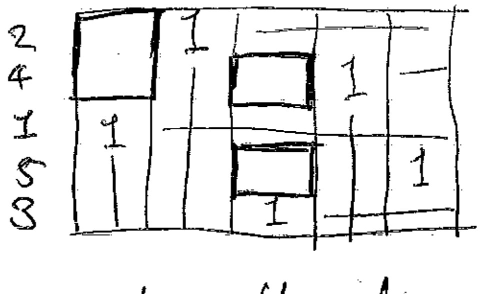
" " " x_2, " " "

" " " x_1.

Lets compute init S_π, from the "bottom pipe dream."

Def^n: The diagram of a perm $\pi = \begin{bmatrix} & & \\ & & \\ & & \end{bmatrix}$ is the boxes left over after crossing out ~~S & E~~ of south and east lines from the 1s.

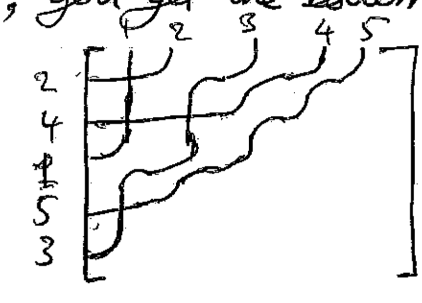
i.e.



Exercise: $|\mathrm{diag}(\pi)| = \ell(\pi)$.

Prop^n ① If you shove the diagram leftward and ~~let~~ turn it into +s, you get the bottom pipe dream.

~~Eg~~ I.e.



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② Any arrangement of +s where each row is ++++ ⌐⌐⌐ ... is the bottom pipe dream of a unique π.

Thm: Schubert poly's are a basis.

Pf: To write p, reduce to

$p - cS_\pi$, where init $p = c \cdot$ monomial, and init S_π = that monomial.

and expand $(p - cS_\pi)$ by induction on init.

L.I. (or uniqueness) by applying the ∂_π to $\sum c_\pi S_\pi = 0$ to show $c_\pi = 0$. □

Thm: We have a $\cancel{\mathbb{Z}}$ $\mathbb{Z}$-basis for $\mathbb{Z}[x_1, \dots, x_n]^{S_n}$ given by $\{\mathfrak{S}_\pi : \pi$ has at most one descent, $\pi(d) > \pi(d+1)\}$

Schur polynomials