Symmetric Polys.

We defined ∂_i the divided difference operator.

S_∞, the group of finite perms on $\mathbb{Z}$

Goal: $\exists!$ a $\mathbb{Z}$ basis of $\mathbb{Z}[x_1 \ldots] = \{\mathfrak{S}_\pi : \pi \in S_\infty\}$ homogeneous,

s.t. $\mathfrak{S}_{id} = 1$, $\partial_i \mathfrak{S}_\pi = \begin{cases} 0 \text{ if } \pi(i) < \pi(i+1) & \text{(ascent)} \\ \mathfrak{S}_{\pi \circ (i\,i+1)} \text{ if } \pi(i) > \pi(i+1) & \text{(descent)} \end{cases}$

Def: $W_0^n := n \; n-1 \ldots 1$

Ex: $\mathfrak{S}_{W_0^n}$ will be $\prod_1^n x_i^{n-i}$

$\mathfrak{S}_{(123)}$ $\mathfrak{S}_{321} = x_1^2 x_2$

$\partial_1 \mathfrak{S}_{321} = x_1 x_2$, $\partial_2 \mathfrak{S}_{321} = x_1^2 = \mathfrak{S}_{312}$, $\partial_1 \partial_2 \mathfrak{S}_{132} = x_1 + x_2$

Existence is hard as there are multiple ways to fix σ using these rules.

$\ell(\pi) = \#\{(i,j) \text{ s.t. } i<j, \pi(i) > \pi(j)\}$

Lemma: $\ell(\pi \circ (i \leftrightarrow i+1)) = \ell(\pi) \pm 1$

Pf of uniqueness: If $\pi \neq id$ then $\mathfrak{S}_\pi \neq 0$ and $\deg \mathfrak{S}_\pi > 0$. (degree = # inversions)

Let i be a descent of π. $\partial_i \mathfrak{S}_\pi = \mathfrak{S}_{\pi \cdot (i\, i+1)} \neq 0$ and by induction

$\deg \mathfrak{S}(\pi) = \ell(\pi)$

Now claim $\bigcap_i^n (\ker \partial_i) = \mathbb{Z}$ as if a monomial is in this set then it is infinite ※

Suppose we know $\partial_i \mathfrak{S}_\pi$ $\forall i$, then if we have $\partial_i \mathfrak{S}_{\pi'} = \partial_i \mathfrak{S}_\pi$.

then $\pi = \pi'$ as their difference is homogeneous and $\partial_i(\mathfrak{S}_\pi - \mathfrak{S}_{\pi'}) = 0 \; \forall i$

Dfn: A reduced word Q for $\pi \in S_n$ is a list of numbers

s.t. $\pi = (q_1 \leftrightarrow q_1 + 1) \circ \dots (q_{|Q|} \leftrightarrow q_{|Q|} + 1)$

$|Q|$ is minimal.

Claim: $|Q_\pi| = \ell(\pi)$

Pf: If $\pi = id$ then done

Otherwise $\exists$ d a descent of π

$\ell(\pi) = \ell(\pi \circ (d \;\; d+1)) + 1$

Shorter than π

So $|Q| \leq \ell(\pi)$

Now let Q be reduced and then ~~Q~~ claim any subword of Q is

reduced so let $Q' = Q \setminus$ last letter. then by induction $|Q'| = \ell(\pi Q')$

$= \ell(\pi Q) - 1$

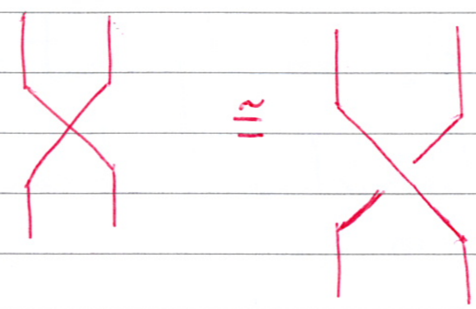
Symmetric Polys

Thm: Any two reduced words for π can be connected by the following moves:

① : $ij \leftrightarrow ji$ for $|i-j|>1$

② : $i\ i{+}1\ i \leftrightarrow i{+}1\ i\ i{+}1$

Braid(n) is a group where we can swap one element over or under another and is similar to S_n.



S_n $\qquad$ Braid_n

Pf: Let $\pi \neq Id$, Q a reduced word for π, m the least number appearing in Q. We replace Q by Q ending in $\pi^{(m)-1}$. We try to move this through the first m rightwards. We now get stuck only on

$m\ m{+}1$ (there are now $m-1$ or m^2 terms)

We now push this rightwards as a block, we now get stuck on

$m \_ m{+}j$ ~~m+j and~~ $m+k$ $\quad k<j$

and then we braid to get

$m \_\ m{+}j\ k\ m{+}k{+}1\ m{+}k \_ m{+}j \rightarrow m \_\ m{+}k{+}1\ m{+}k\ m{+}k{+}1 \_\ m{+}j$

and we can commute the left hand $m+j+1$ out. So the word now ends in $m \dots \pi(m)-1$

We can remove this part of the word from ~~end~~ our original word and by induction on length ~~the~~ any two words are similar under these moves.

eg. $\underline{123}2 \to 1323 \to \cancel{1}3\underline{123}$

Cor: Given π a permutation and Q a reduced word of π.

define $\partial_\pi = \prod_1^{l(\pi)} \partial_{q_i}$ is ~~ass~~ independent of Q

Thm: Schubert polynomials exist and can be computed by the previous process on $\mathfrak{S}_{w_0}$. $\mathfrak{S}(W_0^n) = x_1^{n-1} \dots x_n^0$

So $\mathfrak{S}_\pi = \partial_{\pi^{-1} w_0^n} \mathfrak{S}_{w_n^0}$

Pf: We need to show these are well defined, we need to show the choice of n does not matter.

From ~~Exercise~~ Exercise: Let $\pi \circ \rho = \sigma$, show $l(\pi \cdot \rho) \leq l(\pi) + l(\rho)$

and $\partial_\pi \partial_\rho = \begin{cases} \partial_\sigma & \text{if } = \text{ in inequality} \\ 0 & \text{if } < \end{cases}$

Check well definedness for W_0^n in S_{n+1}

$\therefore$ want $\partial_{W_0^n} \partial_{W_0^{n+1}} \mathfrak{S}_{W_0^{n+1}} = \mathfrak{S}_{W_0^n}$

LHS: $\partial_n \partial_{n-1} \dots \partial_1 \mathfrak{S}_{W_0^{n+1}} = \prod (\partial_i) (x_1^{n-1} x_2^{n-2} \dots x_{n-1})$

Now $\partial_1 (x_1^{n-1} \ldots x_n^0) = x_1^{n-2} x_2^{n-2} \ldots x_n^0$

$\partial_2 (x_1^{n-2} x_2^{n-2} \ldots x_n^0) = x_1^{n-2} x_2^{n-3} x_3^{n-4} \ldots x_n^0$

$\therefore$ result holds as

Fact: If Q is a reduced word for $\pi^{-1} W_0^n$ then

$Q\ n\ n-1\ n-2 \ldots 1$ is reduced for $\pi^{-1} W_0^{n+1}$ in S_n.

$\therefore \partial_{W_0^{n-1} W_0^{n+1}} \mathfrak{S}_{W_0^{n+1}} = \mathfrak{S}_{W_0^n}$

So $\mathfrak{S}_{W_0^n}$ has the same value in S_{n+1} and by induction in S_m, $m \geq n$

therefore our definition is well defined.

It is now trivial to show

$$\partial_i \mathfrak{S}_\pi = \begin{cases} 0 & \text{if } i \text{ is a } \pi\text{-ascent} \\ \mathfrak{S}_{\pi.(i\,i+1)} & \text{if } i \text{ is a } \pi \text{ descent.} \end{cases}$$

Lemma: If $l(\pi) = l(\rho)$ then $\partial_\pi \mathfrak{S}_\rho = \begin{cases} 1 & \text{if } \pi = \rho \\ 0 & \pi \neq \rho \end{cases}$

Pf: If $l(\pi) = l(\rho) = 0$, $\pi = \rho = id$ ✓

Otherwise $\exists$ a descent i of π $\therefore \exists_a$ reduced word for π $Q\,i$
then $\partial_\pi \mathfrak{S}_\rho = \partial_{\pi.(i\,i+1)} \partial_i \mathfrak{S}_\rho = 0$ unless ρ has a descent at i.
So:

$$\partial_\pi \mathfrak{S}_\rho = \partial_{\pi.(i\,i+1)} \mathfrak{S}_{\rho.(i\,i+1)} \quad \text{if descent}$$

and by induction on $l(\pi)$ this is 1 or 0 appropriately