Probabilistic method

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Assignment 1

Problem 1: Suppose $n \geq 4$ and let H be an n-uniform hypergraph with at most $4^{n-1}/3^n$ edges. Prove that there is a coloring of the vertices of H by four colors so that in every edge all four colors are represented.

Problem 2: Let $\mathcal{F}$ be a finite collection of binary strings of finite length and assume that no member of F is a prefix of another. Let N_i denote the number of strings of length i in $\mathcal{F}$. Prove that $\sum_i N_i 2^{-i} \leq 1$.

Problem 3: Let $\{(A_i, B_i), 1 \leq i \leq h\}$ be a family of pairs of subsets of the set of integers such that $|A_i| = k$ for all i and $|B_i| = l$ for all i, $A_i \cap B_i = \emptyset$ and both $A_i \cap B_j, A_j \cap B_i \neq \emptyset$ for all $i \neq j$. Prove that $h \leq \binom{k+l}{k}$.
Hint. Take random ordering of ground set.

Problem 4: Let T be a tournament (complete graph whose edges are oriented) on n vertices such that for every subset of vertices U of size k there is a vertex v which dominates U, i.e., T contains all directed arcs $\{(v, u) : \ u \in U\}$.
(a) Prove that for some positive constant c there exists such a tournament on n vertices , with $n \leq ck^2 2^k$.
(b) Prove that $n > 2^k$.
(c) (**hard, bonus**) Prove that $n > ck2^k$, for some positive constant c.

Problem 5: Prove that there is an absolute constant $c > 0$ with the following property. Let A be an n by n matrix with pairwise distinct entries. Then there is a permutation of the rows of A so that no column in the permuted matrix contains an increasing sub-sequence of length at least $c\sqrt{n}$.

Problem 6: Let $n > 1$ and let $G = (V, E)$ be a bipartite graph on 2^n vertices with a list $S(v)$ of at least n colors associated with each vertex $v \in V$. Prove that there is a proper coloring of G assigning to each vertex v a color from its list $S(v)$.