Probabilistic method

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Assignment 2

Warm up problem: Suppose $n \geq 2$ and let H is an n-uniform hypergraph with 4^{n-1} edges. Show that there is a coloring of H by four colors so that no edge is monochromatic.

Problem 1: Prove that there is a positive constant c so that every set A of n nonzero reals contains a subset $B \subset A$ of size $|B| \geq cn$ so that there are no $b_1, b_2, b_3, b_4 \in B$ satisfying $b_1 + 2b_2 = 2b_3 + 2b_4$.

Problem 2: (a) Prove by induction that every tournament T on n vertices contains a Hamiltonian path (i.e., directed path that goes through every vertex once).
(b) Prove that there exists a tournament T on n vertices which contains at least $n!2^{-(n-1)}$ distinct Hamiltonian paths.

Problem 3: Let $v_1, v_2, \ldots, v_n$ be n vectors in R^n, each of Euclidean norm at most 1, and let $u = \sum_{i=1}^n p_i v_i$, where $0 \leq p_i \leq 1$ for all i.
(i) Prove that there are $\epsilon_i \in \{0, 1\}$ such that

$$||\sum_{i=1}^{n} \epsilon_i v_i - u|| \leq \sqrt{n}/2.$$

(ii) Prove that the above estimate is tight for all n.
(iii) (**hard, bonus**) Prove that even for $m > n$ and for $v_1, \ldots, v_m \in R^n$, each of norm at most 1, and for $u = \sum_{i=1}^m p_i v_i$ with $0 \leq p_i \leq 1$, there are $\epsilon_i \in \{0, 1\}$ such that

$$||\sum_{i=1}^{m} \epsilon_i v_i - u|| \leq \sqrt{n}/2.$$

Problem 4: Let G be a graph on n vertices with minimum degree $d > 1$. Show that G has dominating set of size at most $n\frac{1+\ln(d+1)}{d+1}$. A dominating set of a graph G is a subset of vertices U such that every vertex v of G is either in U or has a neighbor in U.

Problem 5: Prove that every 3-uniform hypergraph with n vertices and $m \geq n/3$ edges contains an independent set (i.e., set with no edges inside) of size at least $\frac{2n^{3/2}}{3\sqrt{3}\sqrt{m}}$.