Probabilistic method

Instructor: Benny Sudakov

Assignment 3

Problem 1: Let G_1 and G_2 be two graphs on the same vertex set V and suppose that G_i has m_i edges, for $i = 1, 2$. Use second moment to prove that there is a constant c and a partition of V into two classes A and B so that for both $i = 1, 2$ we have at least $m_i/2 - c\sqrt{m_i}$ edges of G_i from A to B.

Problem 2: Let X be a random variable taking integral nonnegative values, let $E(X^2)$ denote the expectation of its square, and let $Var(X)$ denote its variance. Prove that

$$Prob(X = 0) \leq \frac{Var(X)}{E(X^2)}.$$

Problem 3: Let $G = (V, E)$ be a simple graph with n vertices and m edges and let λ be the integer. Prove that

(i) There are at least $\lambda^n(1 - m/\lambda)$ proper vertex colorings of G with λ colors.

(ii) The number of proper vertex colorings of G with λ colors is at most $\lambda^n(\lambda - 1)/m$.

(iii) Show that the upper bound from part (ii) can be improved further to $\lambda^n \frac{\lambda-1}{\lambda+m-1}$

Problem 4: Let $v_1 = (x_1, y_1), \ldots, v_n = (x_n, y_n)$ be n two-dimensional vectors, where each x_i and each y_i is an integer whose absolute value does not exceed $\frac{2^{n/2}}{100\sqrt{n}}$. Show that there are two disjoint sets $I, J \subset \{1, 2, \ldots, n\}$ such that

$$\sum_{i \in I} v_i = \sum_{j \in J} v_j.$$

Problem 5: (**hard**) Prove that for every set X of at least $4k^2$ distinct residue classes modulo a prime p, there is an integer a such that the set $\{ax \ (mod\ p) : x \in X\}$ intersects every interval of length at least p/k in $\{0, 1, \ldots, p-1\}$.

Hint. Pick random residues a and b and consider $\{ax + b \ (mod\ p) : x \in X\}$, use now second moment.