Probabiliste Method by Benny Sudakov

Th: $\forall k, l \in \mathbb{N}\ \exists N \in \mathbb{N}$ s.t. every R/B-edge coloring of K_n contains:

1) red clique of size k

or

2) blue clique of size l

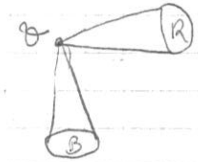
Def.: $R(k,l) :=$ min such N

ex: $R(3,3) = 6$

Prop: $R(k,l) \leq \binom{k+l-2}{k-1} = N$

Proof: (by induction on $k+l$)

suppose we have a R/B-edge coloring of K_n with no red clique of size k and no blue clique of size l



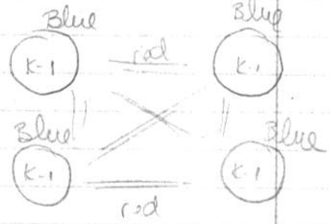
$$|R| \leq \binom{k-1+l-2}{k-2} - 1$$

$$|B| \leq \binom{k+(l-1)-2}{k-1} - 1$$

$$N = |R| + |B| + 1 \leq \underbrace{\binom{k-1+l-2}{k-2} + \binom{k+(l-1)-2}{k-1}}_{=\binom{k+l-2}{k-1}} - 1$$

Cor: $R(K,K) \le \binom{2K-2}{K-1} \le 2^{2K}$

Note: $R(K,K) \ge (K-1)^2$



Th: $R(K,K) \geq 2^{K/2}$

Proof: $N = 2^{K/2}$ and $\forall (i,j)$

color edge (i,j) $\to$ red with probability $1/2$

$\to$ Blue

$\forall$ set S : $|S| = K$, $P[S \text{ monochrom}] = 2 \cdot \frac{1}{2^{\binom{K}{2}}}$

$= \frac{1}{2^{\binom{K}{2}-1}}$

$P[\exists$ a monoch. set of size $K] \le$

$$\le \binom{N}{K} \cdot \frac{1}{2^{\binom{K}{2}-1}} \le \frac{N^K}{K!} \cdot \frac{2}{2^{K\frac{(K-1)}{2}}} < 1$$

$$= \frac{2^{K^2/2}}{K!} \cdot \frac{2}{2^{\frac{K^2}{2}}} \cdot 2^{K/2} \longrightarrow 0$$

Question: Let $\mathcal{F}$ be a collection of subsets of $[n]$ s.t. $\forall F, F' \in \mathcal{F}$

$F \not\subset F'$, $\max |\mathcal{F}| = ?$

ex: All sets of size $\lfloor \frac{n}{2} \rfloor$ or $\lceil \frac{n}{2} \rceil$

give $\max |\mathcal{F}| \geq \binom{n}{\frac{n}{2}}$

Th: Any such $\mathcal{F}$, has $|\mathcal{F}| \le \binom{n}{\frac{n}{2}}$

Proof: claim $\sum_{F \in \mathcal{F}} \frac{1}{\binom{n}{|F|}} \le 1$

$$|\mathcal{F}| \cdot \frac{1}{\binom{n}{\lfloor \frac{n}{2} \rfloor}} \le \sum_{F \in \mathcal{F}} \frac{1}{\binom{n}{|F|}} \le 1$$

Take a random ordering of the grand set $\pi(1), \pi(2), \dots, \pi(n)$

$\forall F \in \mathcal{F}$, $|F| = t$, let

$A_F = \{$ that $\{\pi(1), \pi(2), \dots, \pi(t)\} = F \}$

claim 1: $A_F \cap A_{F'} = \emptyset$; for $F \neq F' \in \mathcal{F}$

$\Updownarrow$

$F \not\subset F'$, $F' \not\subset F$

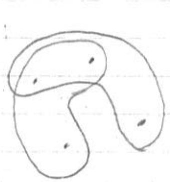
Cor: $\sum_{F \in \mathcal{F}} P[A_F] \leq 1$.

Pr: $P[A_F] = \frac{|F|!\,(n-|F|)!}{n!} = \frac{1}{\binom{n}{|F|}} \leq 1$

Cor: $\forall a_1, \dots, a_n \in \mathbb{R}$ s.t $|a_i| \geq 1$ and $\forall a,b$, $\#\{\varepsilon_i \text{ s.t. } \sum_{i=1}^{n} \varepsilon_i a_i \in]a,b[\}_{b-a \leq 2} \leq \binom{n}{\lfloor \frac{n}{2} \rfloor}$

Def: H is n-uniform hypergraph if all edges have size n

ex.:



Def: n-uniform hypergraph H is 2-colorable if you can color vertices R/B s.t. no edge is monochromatic

ex.: Δ not 2-colorable

Def: $m(n) = \min \#$ of edges in n-uniform non-2-colorable hypergraph

Th: $m(n) > 2^{n-1}$

Proof Let H be n-uniform with edges $e_1, e_2, \dots, e_m$

Color every vertex from the ground set R/B with probability $1/2$ independently

Let define $A_{e_i} := \{\text{event that edge } e_i \text{ is monochromatic}\}$

then $P[A_{e_i}] = 2 \cdot \frac{1}{2^n} = \frac{1}{2^{n-1}}$

$P[\text{some edge is monochromatic}] < \sum_{i=1}^{m} P[A_{e_i}] = m \cdot \frac{1}{2^{n-1}} = 1$

events not-disjoint

In fact, we know that

$2^{n-1} < m(n) < 2 \cdot n^2 \cdot 2^n$

probabilistic $\longrightarrow$

→ construction Take the universe of size n^2. Take $e_1, e_2, \dots, e_m$. $m = 2 \cdot n^2 \cdot 2^n$ independent subsets of size n.

Nati Linial → Combinatorics and geometry

Kahn Kalai's refutation of the Borsuk conjecture

Def: the Diameter of a set $S \subseteq \mathbb{R}^d$ is $\mathrm{diam}(S) := \sup_{x,y \in S} \mathrm{dist}(x,y)$

→ Borsuk asked: Let $S \subseteq \mathbb{R}^d$ s.t. $\mathrm{diam}(S) = 1$. We'd like to cover S by a few open sets each of diameter < 1.

ex: S

dim = 2
\# parts = 3

dim = 1

Easy fact: the ball of diam 1 in $\mathbb{R}^d$ can be split into $d+1$ parts each one of diam < 1

Borsuk-Ulam's Thm: Let $A_1, \dots, A_d$ be open sets s.t. $\bigcup_i A_i = S^{d-1}$, where S^{d-1} is the unite sphere in $\mathbb{R}^d$. then, at least one of the A_i's contains a pair of antipodal points

Borsuk's problem: what is the smallest $f(d)$ s.t. every $S \subseteq \mathbb{R}^d$ of diam 1 can be split into $\leq f(d)$ sets of diam < 1?

Is it true that $f(d) = d+1$?

Some evidence for the conjecture

① $f(2) = 3$

② the conjecture is also true for a "smoth" S