Lecture 2

Probabalistic Method

Expectation

Random variable $X = x_1, \dots, x_k$

$p_1 \quad \dots \quad p_k$ (probabilities)

$\sum p_i = 1$

$\mathbb{E}[X] = \sum_{i=1}^{k} p_i x_i$, a weighted average.

Claim: if $\mathbb{E}[X] = a$, then
(i) $\exists\, x_i \geq a$
(ii) $\exists\, x_j \leq a$

Proof: Suppose all $x_i < a$.
Then $\mathbb{E}[X] = \sum x_i p_i < \sum a p_i = a$ # $\square$

FACT: $\forall$ 2 random variables Z_1, Z_2
$\mathbb{E}[Z_1 + Z_2] = \mathbb{E}[Z_1] + \mathbb{E}[Z_2]$.

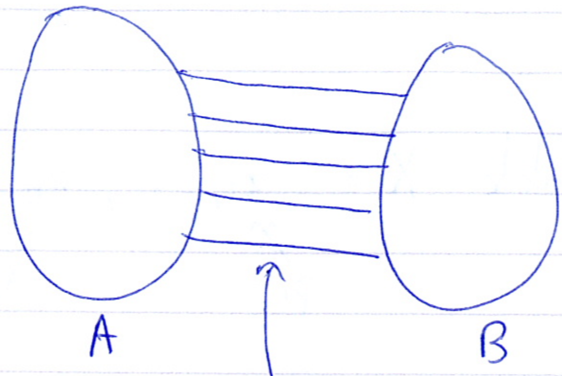
Proof: See any standard probability text, or try it yourself.

Typical application: Z (a complicated r.v.)
$= \sum_{i=1}^{m} Z_i$ where $Z_i = \begin{cases} 1 \\ 0 \end{cases}$, so $\mathbb{E}[Z_i] = \mathbb{P}[Z_i = 1]$.

Hence, by linearity of expectation, $\mathbb{E}[Z] = \sum_{i=1}^{m} \mathbb{E}[Z_i]$.

e.g G = graph, with m edges.

GOAL:



maximise the number of those edges (find a large cut).

cut ≡ splitting

Claim: every graph with m edges contains a cut with $\geq \frac{m}{2}$ edges.

Proof: split vertices randomly.

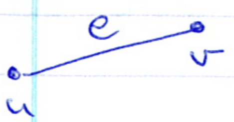
$$\mathbb{P}[v \in A] = \mathbb{P}[v \in B] = \frac{1}{2}.$$

Let $Z \equiv \#$ of edges between A & B.

GOAL: $\mathbb{E}[Z] \geq \frac{m}{2}$.

Write $Z = \sum_{e \equiv \text{edge}} Z_e$, where $Z_e = \begin{cases} 1 & \text{if } e \equiv \text{edge is across from } A \text{ to } B \\ 0 & \text{else} \end{cases}$

Now $\mathbb{E}[Z_e] = \mathbb{P}[\text{edge } e \text{ has one point in } A \text{ \& one in } B]$



$$= \mathbb{P}\left[\begin{matrix} u \in A \\ v \in B \end{matrix}\right] + \mathbb{P}\left[\begin{matrix} u \in B \\ v \in A \end{matrix}\right]$$

$$= \frac{1}{4} + \frac{1}{4}$$

$$= \frac{1}{2}$$ since vertices assigned independently.

So $\mathbb{E}[Z] = \sum_{e = \text{edge}} \frac{1}{2}$

$= \frac{m}{2}$. So by previous comment, there is a cut with $\geq \frac{m}{2}$ edges.

$\square$.

Q:- $A \subseteq \mathbb{Z}$, $|A| = n$.
Find maximal $B \subseteq A$ s.t. no $x, y, z \in B$ satisfy $x + y = z$.

[B is called a 'sum-free' set]

This question was asked by Erdős in the 1960s.

Claim: $\forall A \subseteq \mathbb{Z}$, $|A| = n$ contains a sum-free set of size $> \frac{n}{3}$.

Proof: • Choose large prime p s.t. all your numbers are in absolute values $< p$. Moreover, let $p = 3k + 2$.
• Now view $A \subset \mathbb{Z}_p$ (choose p bigger than 3 times anything in A).

• Now look at $\{k+1, k+2, \ldots, 2k+1\} \equiv$ sum free.
Why? Smallest sum is $2(k+1) = 2k+2$, too big
Largest sum is $2(2k+1) = 4k+2 = p + k = k$, too small.

• The size of this set is $k+1$ (over a third of $\mathbb{Z}_p$).

Let $x \in \mathbb{Z}_p, x \neq 0$. Consider $x \cdot A = \{x \cdot a \mid a \in A\}$, shift.

Claim: Look on $B = \{a \in A \mid x \cdot a \in I\}$, I the middle third. Then B is sum-free.

Proof of claim: Suppose $\exists\, b_1, b_2, b_3 \in B$ with $b_1 + b_2 = b_3$

$xb_1, xb_2, xb_3 \in I$ by definition

$\therefore\ xb_1 + xb_2 = xb_3$ $\times$.

Now, take $x \in \mathbb{Z}_p$ at random (including 0).
Let $Z = |xA \cap I|$ a random variable.
If $\mathbb{E}[Z] > \frac{n}{3}$, then we would be done. ✓

Write ~~$Z = \sum_{i=0}^{p-1} Z_i$, where $Z_i = \{$~~

$$Z = \sum_{a \in A} Z_a, \quad \text{where } Z_a = \begin{cases} 1 & x.a \in I \\ 0 & \text{else} \end{cases}$$

Then $\mathbb{E}[Z] = \sum_{a \in A} \mathbb{E}[Z_a]$

$$\mathbb{E}[Z_a] = \mathbb{P}[Z_a = 1] = \mathbb{P}[a.x \in I] = \frac{|I|}{p}$$

$\boxed{\text{Assume } 0 \notin A}$

(Since $a.x$ is a uniform element of $\mathbb{Z}_p$.

$$= \frac{k+1}{3k+2} > \frac{1}{3}.$$

Hence $\mathbb{E}[Z] > \frac{n}{3}$. $\square$

So we proved here that $\exists$ sum-free $B \subseteq A$ with $|B| \geq \frac{n+1}{3}$.

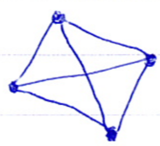
Using much more sophisticated machinery you can improve to $\frac{n+2}{3}$.

Open - how much can you improve?
The problem is that once you start considering the second moment, the dependence of the Z_a becomes important.

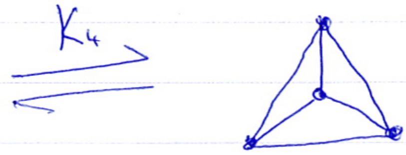
Planar graphs

Definition: G is planar if it can be drawn on the plane with no edge crossings.

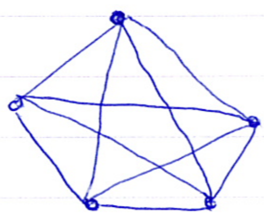
e.g. Yes



K_4

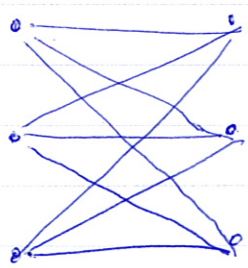


No



$= K_5$ (see problem set for Combinatorics & Geometry for proof).

No $K_{3,3}$



Q: Given G, how many crossings $cr(G)$ has it's drawing on the plane (find minimum)?

Baby Q: Suppose you take $G = K_n$
$cr(K_n) = ?$
We might expect $\approx n^4$ crossings, since roughly every edge will intersect every other edge.

L2 PM

Theorem: If G has n vertices and m edges with $m \geqslant 4n$ then

$$cr(G) \geqslant \frac{1}{64} \frac{m^3}{n^2}.$$