Sudakov

Random Variable $\quad X = x_1, \dots, x_k$ with $x_1 \to p_1$, …, $x_k \to p_k$

$\sum_i p_i = 1$

$\mathbb{E}[X] = \sum_{i=1}^{K} p_i x_i$

Claim: if $\mathbb{E}[X] = a$

(i) $\exists\, x_i \geq a$

(ii) $\exists\, x_i \leq a$

Pf: Suppose ~~th~~ all $x_i < a$

$\mathbb{E}[X] = \sum_i x_i p_i < a \sum_i p_i < a$. Contradiction

Fact: $\forall$ 2 random variables Z_1, Z_2

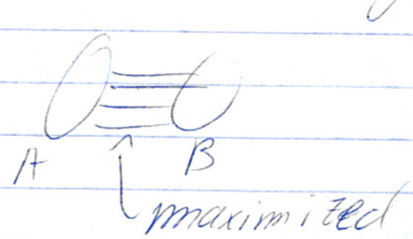
$\mathbb{E}[X_1 + X_2] = \mathbb{E}[X_1] + \mathbb{E}[X_2]$

Typical Application $Z = \sum_{i=1}^{m} Z_i$, $Z_i = \begin{cases} 1 \\ 0 \end{cases}$ takes values

$\mathbb{E}[Z_i] = \Pr[Z_i = 1] \Rightarrow E[Z] = \sum_{i=1}^{m} \mathbb{E}[Z_i]$

$G \equiv$ graph, with m edges

Goal:



Claim: Every graph with m edges contains a cut with $\geq \frac{m}{2}$ edges

Pf: Split vertices randomly

$\Pr[v \in A] = \Pr[v \in B] = \frac{1}{2}$

Let z = # edges between A & B

Goal: $\mathbb{E}[z] \geq \frac{m}{2}$

$z = \sum_{e = edge} z_e$ $\qquad z_e = \begin{cases} 1 & \text{if } e = \text{edge is across} \\ 0 \end{cases}$

$\mathbb{E}[z_e] = \Pr[$ edge has one vertex in A and one in $B]$

$\Pr[e \text{ is across}] = \Pr\begin{bmatrix} u \to A \\ v \to B \end{bmatrix} + \Pr\begin{bmatrix} u \to B \\ v \to A \end{bmatrix}$

$u \xrightarrow{e} v$ $\qquad = \frac{1}{2} \cdot \frac{1}{2} + \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{2}$

Q: $A \subseteq \mathbb{Z}$, $|A| = n$ Sum-free

find maximum $B \subseteq A$ s.t. no $x, y, z \in B$

Claim: $\forall A \subseteq \mathbb{Z}$, $|A| = n$
Contains sum-free set of size $\geq \frac{n}{3}$

Choose large prime p s.t. all your numbers $|\,| < p$, $p = 3K+2$

$A \subset \mathbb{Z}_p = \{0, 1, 2, \dots, 3K, 3K+1\}$

look on $I = \{K+1, K+2, \dots, 2K+1\}$ = sum free

$\# I = K+1$

$2 \cdot (2K+1) = 4K+2 = p+K = K$

Let $x \in \mathbb{Z}_p$ $x \neq 0$

Consider $x \cdot A = \{x \cdot a \mid a \in A\}$ → shift

look on $B = \{a \in A \mid x \cdot a \in I\} \subset A$

Claim: B is sum-free.

Pf: Suppose $\exists\ b_1, b_2, b_3 \in B$ $b_1 + b_2 = b_3$

$xb_1, xb_2, xb_3 \in I$

$xb_1 + xb_2 = xb_3$ Contradiction.

Take $x \in \mathbb{Z}_p$ at random

Let $Z = |xA \cap I|$ rand. var.

if $\mathbb{E}[Z] > \frac{n}{3}$ ✓

$Z = \sum_{a \in A} Z_a$ $\qquad Z_a = \begin{cases} 1 & x \cdot a \in I \\ 0 & \end{cases}$

$\mathbb{E}[Z] = \sum_{a \in A} \mathbb{E}[Z_a] > \frac{n}{3}$ $\qquad 0 \notin A$

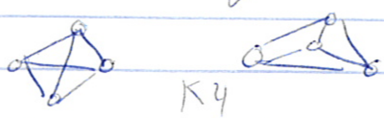
$\mathbb{E}[Z_a] = \Pr[Z_a = 1] = \Pr[a \cdot x \in I] = \frac{|I|}{p} = \frac{k+1}{3k+2} > \frac{1}{3}$

$a \cdot x$ is uniform element of $\mathbb{Z}_p$.

Planar Graphs

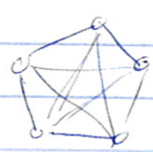
Def: G is planar if it can be drawn on the plane with no edge crossings

Ex: Yes



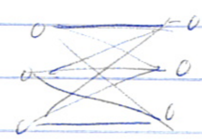
K_4

no



$= K_5$

no



$K_{3,3}$

Q: Given G, how many crossings $Cr(G)$ has it's drawing on the plane (find minimum).

Q: Suppose you take $G = K_n$, $Cr(K_n) = ?$

Th: If G has n vertices and m ~~vertices~~ edges then $Cr(G) \geq \frac{1}{64} \frac{m^3}{n^2}$