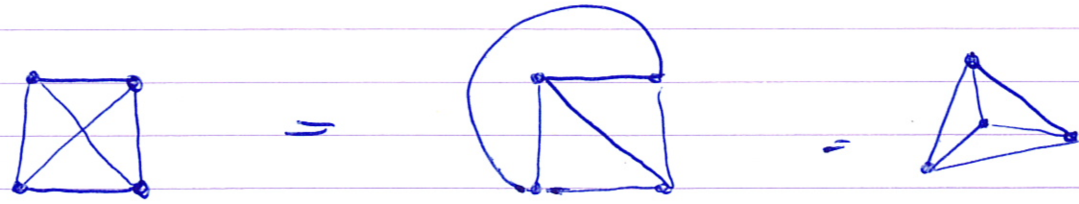
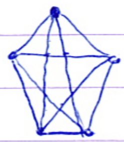
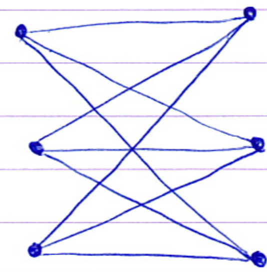
Ex:

Planar:



Not planar:

$K_5 =$ 

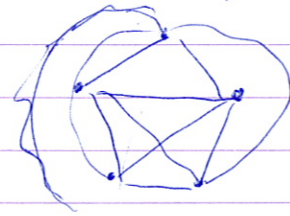
$K_{3,3} =$ 

Q: Given G, how many crossings $cr(G)$ has its drawing on the plane (i.e. $cr(G)$ is the minimal number of crossings)

Q: Suppose you take $G = K_n$. Then $Cr(K_n) = ?$

$(\; Cr(K_1) = Cr(K_2) = Cr(K_3) = Cr(K_4) = 0$

$Cr(K_5) = ?$



Th

If G has n vertices and m edges, $m > 4n$, then

$$Cr(G) \geq \frac{1}{64} \frac{m^3}{n^2}$$

Lesson 3 BENNY SUDAKOV LECTURE 3

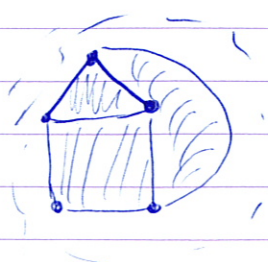
Th

$\forall$ planar drawing of G (G connected).

Then

$$|V| - |E| + |F| = 2$$

Def.



$|V| = 7 \qquad |E| = 7 \qquad |F| = 4$

$5 - 7 + 4 = 2$

Prob Method

- Convex polygons also satisfy this formula (punch a hole in the polygon to get a planar graph).

Cor
Every planar graph G satisfies $|E| \leq 3|V| - 6$

Cor
If G has n vertices, m edges, then

$$Cr(G) \geqslant m - (3n-6) \geqslant m - 3n$$

Th
If G has n vertices, m edges and $m \geqslant 4n$, then

$$Cr(G) \geqslant \frac{1}{64} \frac{m^3}{n^2}$$

Proof: Start with a drawing of G which minimizes the number of crossings. Pick each vertex of G independently, with probability p (p small, to be chosen later), so we get a subset $V' \subset V(G)$ s.t. each vertex $v \in V(G)$ belongs to V' with probability p. (vertices of G)
Let $E' \subset E(G)$ (edges of G) be the set of edges with both endpoints in V'.

Let G' = graph with vertex set V' and edge set E'.

We know $Cr(G') \geqslant |E'| - 3|V'|$.
So $E[Cr(G')] \geqslant E[|E'|] - 3E[|V'|]$

$$E[|V'|] = \cancel{}\, n \cdot p \quad \Big(= \sum_{v \in V} \overbrace{P(v \in V')}^{=p} \Big)$$

$$E[|E'|] = m \cdot p^2 \quad \Big(= \sum_{e \in E} \underbrace{P(e \in E')}_{=p^2} \Big)$$

(This is not the crossing number of G', it's the number of crossings in our fixed drawing of G.)

So $E[Cr(G')] =$
$= p^4\, Cr(G)$ (each crossing survives with probability p^4.)

So we get $p^4 \, Cr(G) \geq m p^2 - 3pn$

$$\Leftrightarrow \quad Cr(G) \geq m p^{-2} - 3 n p^{-3}$$

We optimize the inequality by taking $p = \frac{4n}{m}$.

this is why we need $4n \leq m$, because p is a probability.

we get

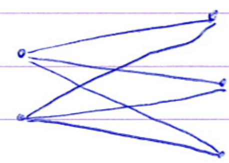
$$Cr(G) \geq m\left(\frac{m}{4n}\right)^2 - 3n\left(\frac{m}{4n}\right)^3$$

$$= \frac{1}{16}\frac{m^3}{n^2} - \frac{3}{64}\frac{m^3}{n^2} = \frac{1}{64}\frac{m^3}{n^2}$$

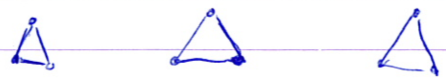
$\square$

Problem (Erdös): $P \subseteq \mathbb{R}^2$ set of n points. How many unit distances can P have?

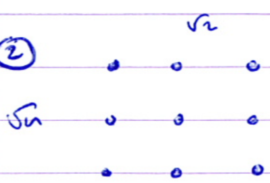
Fact: The unit distance graph in the plane does not contain a $K_{2,3}$



Example:

1.  ...

~~on~~ $\frac{n}{3} \to n$ unit distances, ~~there~~

2. 

Conj

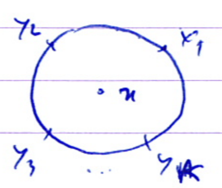
$\#$ of unit distances $\leq n^{1+\varepsilon}$ $\forall \varepsilon > 0$, n large.

Thm

$\#$ of unit distances $\leq c n^{4/3}$

Pf:

Start with P, $|P| = n$ points. Draw a unit circle around each point.



Graph G: Vertices $= P$

Edges $=$ arcs of circles.

$\{y_1, \dots, y_k\} = P \cap$ circle

~~We can assume each point is in at least two~~

We have $Cr(G) \geq \frac{1}{64} \frac{m^3}{n^2}$

~~WLOG~~ $m = ?$ Every vertex participating in k unit distances contributes with k edges.

So $m = 2 \underbrace{(\# \text{ of unit distances})}_{:= z} = 2z$

So $Cr(G) \geq \frac{1}{64} \frac{8z^3}{n^2} = \frac{1}{8} \frac{z^3}{n^2}$

Also, $2\binom{n}{2} \geq Cr(G)$ because ~~each~~ two circles intersect in ~~at~~ at most 2 points.

So

$$\frac{1}{8} \frac{z^3}{n^2} \leq 2\binom{n}{2} \sim 4n^2$$

$\Rightarrow z^3 \leq 8n^4 \Rightarrow z \leq c\, n^{4/3}$ □

__Th__

Let P = set of points in the plane.
L = set of lines in the plane.

Then # of incidences point-line $\leq c(|P| \cdot |L|)^{2/3} + 100|P| + 100|L|$

- Take $A \subseteq \mathbb{R}$, $|A| = n$.

Consider $A + A = \{a + a' \mid a, a' \in A\}$

$|A+A|$ can be linear in n. (Corresponds to arithmetic progressions)

$|A \cdot A|$ can be linear in n. (Corresponds to geometric progression)

<u>Conj.</u> : $\max(|A+A|, |A\cdot A|) \geq n^{2-\varepsilon}$ $\forall \varepsilon$

(i.e. $|A+A|$ and $|A\cdot A|$ cannot ~~ah~~ both be unusually small, because ~~the~~ A can't look like an arithmetic progression and a geometric progression at the same time)

- Using line/point incidence, one can show that

$$\max(|A+A|, |A\cdot A|) \geq n^{5/4}$$

(By elementary methods, it is hard even to show that

$$\max(|A+A|, |A\cdot A|) \geq n^{1+\varepsilon}$$

, for $\varepsilon>0$)

<u>Best result today</u> : $\max(|A+A|, |A\cdot A|) \geq n^{4/3}$