Problem set II

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1. **A little about planar graphs:**

   - Show that the Euler formula $v - e + f = 2$ holds for every connected planar graph G with v vertices, e edges and where a drawing of G in the plane has f faces.
   - Conclude that in a planar graph $e \leq 3v - 6$.
   - Conclude that K_5 is non-planar.
   - Show that the complete bipartite graph $K_{3,3}$ is not planar.
   - Show that every finite planar graph has a vertex of degree ≤ 5. What can you say about the smallest vertex degree in an infinite planar graph?
   - Show that every planar graph is 6-colorable. (Hint: Based on the previous item, find a vertex of degree ≤ 5, now use induction).

2. **Regular graphs can have arbitrarily high girth:**
   In Sudakov's class you saw that graphs can have both arbitrarily high girth and chromatic number. However, the graphs created in that proof are not regular. Here we want to give a somewhat strange proof that it is possible to be regular *and* have arbitrarily high girth. We start from a graph $G = (V, E)$ that is d-regular, has girth g and has exactly T cycles of length g. Our plan is to construct another graph $G' = (V', E')$ whose girth is $\geq g$ and has *strictly* fewer than T cycles of length g.

   - Construction of G': Let $V' = V \times \{1, 2\}$. We next turn to the edge set E'. For every edge $xy \in E$ in G we introduce a pair of edges in E' as follows: Either the pair $(x, 1)(y, 1)$ and $(x, 2)(y, 2)$ or the pair $(x, 1)(y, 2)$ and $(x, 2)(y, 1)$. The choice between these two options is done by independent coin flips (one coin-toss per each edge $xy \in E$).
   - Show that the girth of G' is always $\geq g$.
   - Let X be the random variable that counts the number of g-cycles in G'. Show that the expectation of X is T.
   - Now find a slight modification of the above construction that will make this expecatation strictly smaller than M and conclude the theorem.

3. **Very short cycles are few:**
   Show that if G has girth g, then there are at most $O(n^4)$ cycles of length g in G. Show that

this bound is tight.

If we also assume that G is d-regular, show tha the upper bound can be improved to $O(d^2n^2)$.

4. **Diameter and girth:**
   Show that if a graph has girth g then its diameter is $\geq \lfloor \frac{g}{2} \rfloor$.