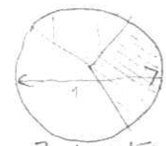
Combinatorics and geometry

Kan Kalai's refutation of the Borsuk conjecture

def: diameter of a set $S \subseteq \mathbb{R}^d$, $\text{diam}(S) = \sup\{d(x,y) : x,y \in S\}$

diam = 1

Borsuk asked: Let $S \subseteq \mathbb{R}^d$, $\text{diam}(S) = 1$.
We'd like to ~~decompose~~ cover S ~~into~~ by a few (open) sets, each of diameter < 1 ?



3 parts

easy fact: the ball of diam 1 in $\mathbb{R}^d$ can be split into $d+1$ parts each of diam < 1

Borsuk-Ulam's thm: Let $A_1, \dots, A_{d+1}$ be open sets s.t. $\cup A_i = S^{d-1}$ a unit sphere in $\mathbb{R}^d$
Then at least one of the A_i's contains a pair of antipodal points.

Borsuk's problem: What is the smallest $f(d)$ st every $S \subseteq \mathbb{R}^d$ of diameter 1 can be split into $\leq f(d)$ sets of diam < 1?

Borsuk's conjecture: $f(d) = d+1$ (false)

It is true that $f(2) = 3$, $f(3) = 4$

The answer $d+1$ is also correct for a "smooth" S.

We will construct a discrete set of points in $\mathbb{R}^d$ of diam 1 such that, if you decompose this set to fewer than $2^{c\sqrt{d}}$ parts, then at least one part still has diam 1.
some absolute constant

0/1 vectors $\longleftrightarrow$ set n $\longleftrightarrow$ charact. vectors of a set

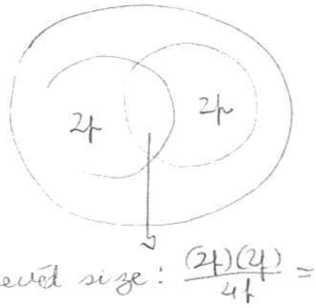
[Sperner's Lemma: If $\mathcal{F}$ is a collection of subset of $[n] = \{1, \dots, n\}$ with no inclusions, $|\mathcal{F}| \leq \binom{n}{\lfloor n/2 \rfloor}$

Erdös-Ko-Rado thm: Let n, k be integers $n > 2k$ and let $\mathcal{F} \subseteq \binom{[n]}{k}$ st $\forall A, B \in \mathcal{F}$ $A \cap B \neq \emptyset$. Then $|\mathcal{F}| \leq \binom{n-1}{k-1}$

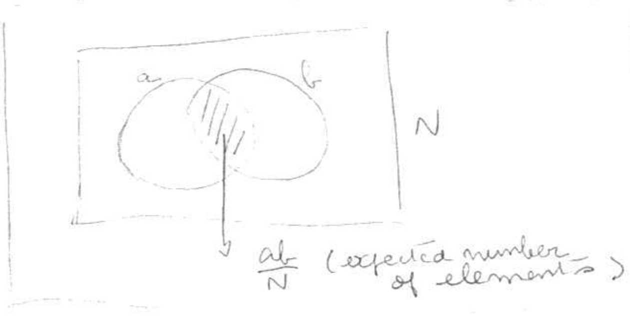
$\left(\binom{[n]}{k} = \{A_i : |A_i| = k, \ A_i \subseteq [n]\} \right)$

Theorem: Let p be a prime and let $\mathcal{F}$ be a collection of sets of cardinality $2p-1$ from $[n]$.
If $|A \cap B| \neq p-1$ $\forall A, B \in \mathcal{F}$ then $|\mathcal{F}| \leq \binom{n}{0} + \binom{n}{1} + \dots + \binom{n}{p-1}$

$n = 4p$



expected size: $\frac{(2p)(2p)}{4p} = p$



$\frac{ab}{N}$ (expected number of elements)

In particular: for $n = 4p$, $|\mathcal{F}| < \dfrac{\binom{n}{p-1}}{1.2^n}$

proof (from theorem):

fact: fix $1 > \alpha > 0$. $\binom{n}{\alpha n} = 2^{(1-\alpha n) n H(\alpha)}$, $H(x) = -x\log_2 x - (1-x)\log_2(1-x)$

$n = 4p$

$\binom{n}{0} + \binom{n}{1} + \dots + \binom{n}{p-1} < \dfrac{\binom{n}{p-1}}{1.2^n} \leftarrow \binom{4p}{p-1} = \dfrac{2^n}{\sqrt{n}}$

$\binom{n}{n/4} \overset{?}{<} \dfrac{2^n}{1.2^n}$

$2^{nH(1/4)} \overset{?}{<} \dfrac{2^n}{1.2^n}$

$2^{H(1/4)} < \dfrac{5}{3}$

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idea for proof of theorem: $\mathcal{F} \to$ collection of vectors in some vector space V

$A \in \mathcal{F} \longleftrightarrow \varphi_A \in V$

1) $\{\varphi_A\}_{A\in\mathcal{F}}$ is linearly independent $\Rightarrow |\mathcal{F}| \leq \dim V < \dots$

2) $\dim V < \dots$

$\psi_A : \langle \varphi_A, \varphi_B \rangle = F_{A,B}$