NATHAN LINIAL - LECTURE 2

Problem: is it true that every ("nice") set of diameter 1 in $\mathbb{R}^d$ can be split into $d+1$ sets of diameter smaller than 1?

Comment: this is true and tight for the unit sphere.

But in general... no!

We are constructing a finite set of points in $\mathbb{R}^{n^2}$ of some diameter Δ such that if S is partitioned into less than 1.14^n subsets, then at least one of the ~~points~~ parts still has diameter Δ.

Theorem: Let p be a prime, let $\mathcal{F} \subseteq \binom{[n]}{2p-1}$ be such that $\forall A, B \in \mathcal{F}$, $|A \cap B| \neq p-1$. Then $|\mathcal{F}| \leq \binom{n}{0} + \dots + \binom{n}{p-1}$.

In particular, if $n = 4p$, then $|\mathcal{F}| < \frac{1}{1.14^n}\binom{4p}{2p-1}$.

Note: If $n=4p$, then we know that if we partition the collection of ~~sets~~ $(2p-1)$-subsets of $[4p]$ into fewer than 1.14^n subclasses, then at least one of them contains two sets with intersection size $p-1$.

Given $A \in \mathcal{F}$, let ~~c_A be~~ $c_A \in \{0,1\}^n$ be the characteristic vector of A.

Plan: Show that $\{c_A\}_{A \in \mathcal{F}} \subseteq V$ is linearly independent and $\dim V \leq \binom{n}{0} + \dots + \binom{n}{p-1}$.

$$A \in \mathcal{F} \longrightarrow c_A \in \{0,1\}^n$$

$$\hookrightarrow f_A : \{0,1\}^n \to \mathbb{F}_p \quad \text{~~(f_A is a linear functional)~~}$$

the critical property is:

$$A, B \in \mathcal{F} \Rightarrow f_A(c_B) = \begin{cases} 0, & B \neq A \\ \neq 0, & B = A \end{cases}$$

Proof of the linear independence of $\{c_A\}_{A \in \mathcal{F}}$:

From $\sum_{B \in \mathcal{F}} \alpha_B c_B = 0$ we get $\sum_{B \in \mathcal{F}} \alpha_B f_A(c_B) = 0$, that is, $\alpha_A f_A(c_A) = 0$, and since $f_A(c_A) \neq 0$, $\alpha_A = 0$. Since $A \in \mathcal{F}$ is arbitrary, the conclusion follows.

Now V is a vector space over $\mathbb{F}_p$. Let $e \in \{0,1\}^n$. Then:

$$\cancel{f_A(e) = \prod_{t=0}^{p-2}\left[\left(\sum_{i\in A} e_i\right) - t\right]} \quad f_A(e) = \prod_{t=0}^{p-2}\left[\left(\sum_{i\in A} e_i\right) - t\right]$$

If $\sum_{i\in A} e_i \equiv_p -1$, then $f_A(e) \neq 0$. Otherwise, $f_A(e) = 0$.

Let $A \neq B \in \mathcal{F}$. By assumption, $|A \cap B| \neq p-1$, and since $0 \leq |A\cap B| \leq 2p-2$, then $f_A(e_B) = 0$, and:

$$f_A(e_A) = \prod_{t=0}^{p-2}(2p-1-t) \not\equiv_p 0$$

~~Let V be a basis~~ V is a subset of the space of all polynomial functions from $\{0,1\}^n$ to $\mathbb{F}_p$.
For $\varepsilon \in \{0,1\}$, $\varepsilon^k = \varepsilon$, so V is a subset of the space of all multilinear functions $\{0,1\} \to \mathbb{F}_p$ of degree less or equal than $p-1$.

<u>Theorem:</u> Let p be a prime and let $n = 4p$. If we partition $\binom{[4p]}{2p-1}$ into fewer than 1.1^n subclasses, then at least one of them contains two sets with intersection size exactly $p-1$.

Given two sets A and B, $\mathrm{dist}(e_A, e_B) = |A \triangle B| = |A\setminus B| + |B \setminus A|$.

<u>Goal:</u> Find a finite set S of points in $\mathbb{R}^{n^2}$ of diameter Δ such that if we partition S into less than 1.1^n subsets, then at least one still has diameter Δ.

$$A \in \binom{[4p]}{2p-1} \rightsquigarrow \begin{array}{c|cc} & A & A^c \\ \hline A & + & - \\ A^c & - & + \end{array} = M_A$$

	$A\cap B$	$A\setminus B$	$B\setminus A$	$(A\cup B)^c$
$A\cap B$	$+^{+}$	$+^{+}$	$+^{-}$	$-^{-}$
$A\setminus B$	$-^{+}$	$+^{+}$	$-^{-}$	$+^{-}$
$B\setminus A$	$+^{-}$	$-^{-}$	$+^{+}$	$-^{+}$
$(A\cup B)^c$	$-^{-}$	$+^{-}$	$-^{+}$	$+^{+}$

$\mathrm{dist}(M_A, M_B) = 2|A\triangle B|(n - |A \triangle B|)$ is largest iff $|A \triangle B| = p-1$