Linial

Borsuk-Ulam Theorem:

If $A_1, \ldots, A_d$ are open sets and $\bigcup A_i = S^{d-1} \Rightarrow$ at least one of A_i's contains an antipodal pair of points.

$G=(V,E)$, K-col of G $\varphi: V \to \{1, \ldots, K\} = [K]$ s.t. $xy \in E \Rightarrow \varphi(x) \neq \varphi(y)$

$\Leftrightarrow$ Partition V into K anticliques

Graph Coloring is an NP-hard question.

Input: $G=(V,E)$, $K \in \mathbb{N}$

Output: Does G have a K-coloring?

Note that if G has n vertices and if $\text{indep}(G) = \ell \Rightarrow \chi(G) \geq \frac{n}{\ell}$

The typical clique # of an n-vertex graph is $O(\log n)$ anticlique.

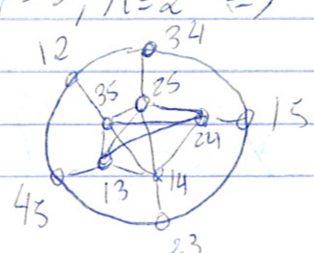
$\Rightarrow$ The typical χ of an n-vertex graph is $\geq \Omega\left(\frac{n}{\log n}\right)$

Kneser Graphs: $K(n,K)$

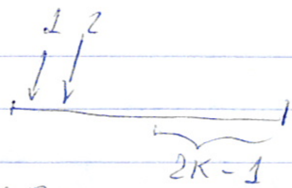
$V = \binom{[n]}{K}$, $AB \in E \Leftrightarrow A \cap B = \emptyset$

$\chi(K(n,K))$?

$n=5, K=2 \Rightarrow K(5,2) =$ Petersen Graph



$C_1 : A \in \binom{[n]}{k} : 1 \in A$



$C_2 : B \in \binom{[n]}{k}$ $1 \notin B,\ 2 \in B$

$\vdots$

$C_t : X \in \binom{[n]}{k}, \min X = t$

$t = 1, \dots, n-2k+1$

$C_{n-2k+2} = \binom{\{n-2k+2, \dots, n\}}{k}$

$$\chi(K(n,k)) \le n-2k+2$$

$$\mathrm{indep}(K(n,k)) = \binom{n-1}{k-1}$$

By Erdős-Ko-Rado

$$\chi(K(n,k)) \ge \frac{\binom{n}{k}}{\binom{n-1}{k-1}} = \frac{n!\,(k-1)!\,(n-k)!}{k!\,(n-k)!\,(n-1)!} =$$

$$= \frac{n}{k}$$

Let G ~~be~~, H be graphs, a graph homomorphism $H \to G$ is a map $\varphi : V(H) \to V(G)$ s.t. if $xy \in E(H)$ $\Rightarrow \varphi(x)\varphi(y) \in E(G)$

We have $\chi(G) \ge \chi(H)$

Rephrasing Borsuk-Ulam

$B(d, \varepsilon)$ is a graph

$V = S^d$, $xy \in E \Leftrightarrow x, y$ are ε-antipodal

$\|x+y\| < \varepsilon$

$\forall \varepsilon > 0$

$\chi(B(d,\varepsilon)) \geq d+2$

We want to find a homomorphism between $K(n,k)$ and $B(d,\varepsilon)$. with $d = n-2k$

Our goal: Find a homomorphism $B(d,\varepsilon) \longrightarrow K(n,k)$, $\varepsilon > 0$, $d = n-2k$

If we can find ~~fn~~ such a homomorphism,
$\Rightarrow \chi(K(n,k)) \geq n-2k+2$

$\leq$ ✓

To every point of $S^d \rightsquigarrow$ set of size k, a subset of $[n]$.

Idea: Can we find n points on S^d s.t. every hemisphere contains at least k of them?

Can we find $n = 2k+1$ points on S^1 s.t. every half-circle contains $\geq k$ of the points?