Graph Theory

Assume Borsuk-Ulam Thm: If $S_1 \dots S_{d+1}$ are ~~open~~ closed sets and $\bigcup S_i = S^{d-1}$ then at least one contains antipodal points.

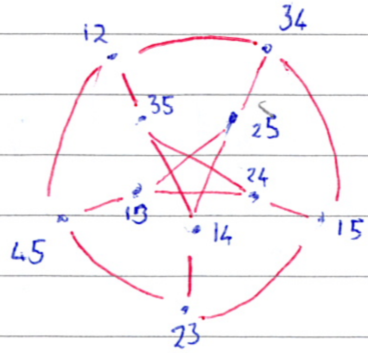
Defn: Given a graph G a k-colouring is a colouring of G in k colours such that no edge has the same two colours. The chromatic number is the least such k. $\chi(G)$

Finding k-colourings is NP-hard.

Note if G has n vertices and $\mathrm{Indep}(G) = \ell$, $\chi(G) \geq n/\ell$.
The typical clique # on n vertices is $\Theta(\log n)$, and the same for anti cliques. The typical χ on n vertices has $\chi(G) \geq \Omega\left(\frac{n}{\log n}\right)$

Kneser graphs: $K(n,k)$. $V = \binom{[n]}{k}$, AB is an edge if $A \cap B = \emptyset$
What is $\chi(K(n,k))$

$n = 5$, $k = 2 \Rightarrow K(5,2)$ Petersen Graph



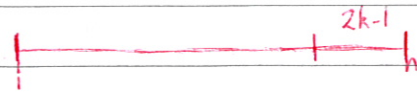
An n-regular graph can be n or $n+1$ coloured, this is the smallest $n+1$ example.

To provide an ~~lower~~ upper bound on $\chi(K(n,k))$ we choose the following colour classes:

C_1: $A \in \binom{[n]}{k}$, $1 \in A$
C_2: $A \in \binom{[n]}{k}$, $2 \in A$, $1 \notin A$
$\vdots$

We stop at $n-2k+1$ and have

$$C_{n-2k+1} = \binom{\{n-2k+2, \dots, n\}}{k}$$



$\therefore$ we have $\chi(K(n,k)) \le n-2k+2$

indep $(K(n,k)) = \binom{n-1}{k-1}$

$$\therefore \chi(K(n,k)) \le \frac{\binom{n}{k}}{\binom{n-1}{k-1}} = \frac{n}{k}$$

Let G, H be graphs, then a graph homom. $\varphi: V(H) \to V(G)$ such that iff v_1 and v_2 are connected then $\varphi(v_1)$ and $\varphi(v_2)$ are connected

if $\varphi: G \to H$ a graph homom. then

$$\chi(G) \le \chi(H)$$

as adding in edges only increases $\chi(G)$ and we can pull back G with extra edges from H

Rephrasing Borsuk-Ulam, $B(d, \varepsilon)$ is a graph: $V = S^d$, $xy \in E$ if x, y are ε antipodal.

Want $\forall \varepsilon > 0$ $\chi(B(d, \varepsilon)) \ge d+2$

Idea Pf: Pick a dense enough finite graph, and find a homomorphism from $B(d, \varepsilon)$ into $K(n,k)$ for $d = n - 2k$. Then our previous bound will work.

So we need to find a mapping from S^d to $\binom{[n]}{k}$

Idea: Can we pick n points such that a hemisphere contains at least k points?