COMBINATORICS AND GEOMETRY

<u>Kneser graphs</u> $K(n,k)$ $\left\{ \begin{array}{l} V = \binom{[n]}{k} \\ AB \in E \iff A \cap B = \emptyset \end{array} \right.$

$\chi(K(n,k)) \leq n-2k+2$

$\chi \geq \frac{\#\text{ vertices}}{\text{indep }\#} = \frac{n}{k}$ → weak lower bound

$\chi \geq \text{clique} = \left\lfloor \frac{n}{k} \right\rfloor$

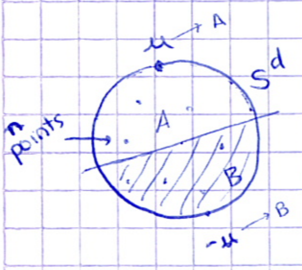
$$\boxed{\exists\ \varphi : H \to G \text{ homomorphism} \implies \chi(G) \geq \chi(H)}$$

we will prove that $\chi(K(n,k)) \geq n-2k+2$ by finding an homomorphism

$$\text{Borsuk}(d,\varepsilon) \longrightarrow \text{Kneser}(n,k)$$

where $d = n-2k$, $\varepsilon > 0$ "small".

<u>Borsuk graphs</u> Borsuk(d,ε) $\left\{ \begin{array}{l} V = S^d \\ \forall x,y \in S^d \\ xy \in E \iff \|x+y\| < \varepsilon \end{array} \right.$



$\exists\ A \in \left\{ \begin{array}{c} n\text{-points set} \\ + \text{hemisphere} \end{array} \right\}$ with at least k points

$\boxed{d=1}$ →
$n = 2k+1$
$S^d = S^1$

Goal: Find n points in $\mathbb{R}^{d+1} \setminus \{0\}$ such that every half-space that is determined by a hyperplane through the origin contains $\geq k$ of the points.

<u>Moment curve</u>

$$\mu : \mathbb{R} \to \mathbb{R}^{d+1}$$
$$t \mapsto (1, t, t^2, \dots, t^d) = \mu(t)$$

$\{ p_j \mid j = 1, \dots, n \}$

$$p_j = (-1)^j \mu(j)$$

$\vec{a} = (a_0, \dots, a_d) \implies$ half-space $H^+ = \{ x \in \mathbb{R}^{d+1} \mid \sum_{i=0}^{d} a_i x_i > 0 \}$

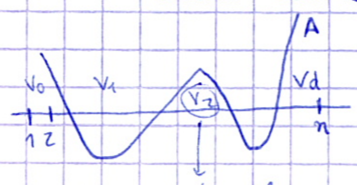
$\mu(t) \in H^+$?

$a_0 + a_1 t + a_2 t^2 + \dots + a_d t^d > 0$

Define A as the polynomial $u \mapsto \sum_{j=0}^{d} a_j u_j$

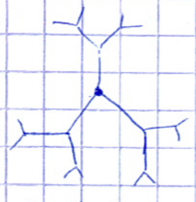
$$\mu(t) \in H^+ \iff A(t) > 0$$

<u>Claim</u> $\forall$ polynomial A of degree $\leq d$ at least k of the inequalities (and at most $k+d$) $(-1)^j A(j) > 0$ hold



$\left\lfloor \frac{V_i}{2} \right\rfloor \leq$ successes in the i-th interval $\leq \left\lceil \frac{V_i}{2} \right\rceil$

Investigating graphs from a geometric perspective



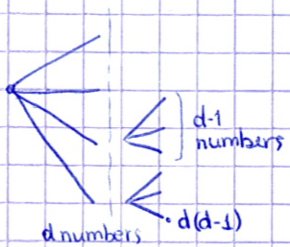
infinite tree

G graph.

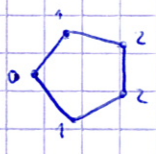
girth G = least length of a cycle in G

Sparsity: want girth large?

$g = 5$, G d-regular
How small can $|V| = n$ be?



Clearly $n \geq 1 + d + d(d-1) = d^2 + 1$.



Q: How many graphs are there
- d-regular
- $n = d^2+1$
- girth $= 5$?

Teo There are either ~~four~~ three or ~~five~~ four values of d for which $\exists G$ d-regular, $n = d^2+1$, $g = 5$.

$d = 2$	5-cycle
$d = 3$	Peterson graph
$d = 7$	Hoffman-Singleton
$d = 57$	? $n = 3250$

Adjacency matrix

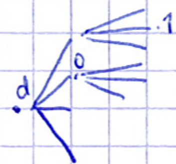
$$G \longmapsto A = \begin{array}{c c} & j \\ i & \begin{bmatrix} 0 & & \\ & 1 & \\ & & 0 \end{bmatrix}_{n\times n} \end{array}$$

$\rightarrow$ it's symmetric

$A_{ij} = 1$ iff $ij \in E(G)$

$$A \cdot \vec{1} = \underbrace{d}_{\text{eigenvalue}} \cdot \vec{1}$$

$\hookrightarrow$ eigenvector



A^2_{ij} = # ways of going from i to j in two stops

$= \sum_k a_{ik} a_{kj}$

$= \# k : \; a_{ik} = a_{kj} = 1$

the adjacency matrix satisfies: the all 1's matrix

$$A^2 = dI + \underbrace{J - A - I}_{\text{all the non-neighbour matrix}}$$

If u is an eigenvector with eigenvalue $\lambda \neq d$

$\Rightarrow \langle u, \vec{1} \rangle = 0$ (because the matrix is symmetric)

$$A^2 u = dIu + Ju - Au - Iu$$
$$\lambda^2 = d + 0 - \lambda - 1$$

$$\lambda^2 + \lambda - (d-1) = 0 \qquad \boxed{\lambda_{1,2} = \frac{-1 \pm \sqrt{4d-3}}{2}}$$

For $\begin{matrix} m_1 \\ m_2 \end{matrix}$ with multiplicity $\begin{matrix} \lambda_1 \\ \lambda_2 \end{matrix}$,

$$\boxed{m_1\lambda_1 + m_2\lambda_2 + d = \cancel{tr} A = 0}$$

$$\boxed{m_1 + m_2 = n - 1 = d^2}$$ (counting multiplicities)

$A = \begin{bmatrix} \ddots & j & j' \end{bmatrix}$ $\quad A^2 = \begin{bmatrix} \ddots & j & j' \end{bmatrix}$ $\quad A + A^2 = \begin{bmatrix} d & \cdots \end{bmatrix}$