<u>Geometry</u> 26/7/12

(Kesner graphs $K(n,k)$, $V = \binom{[n]}{k}$ with $AB \in E \Leftrightarrow A \cap B = \emptyset$)

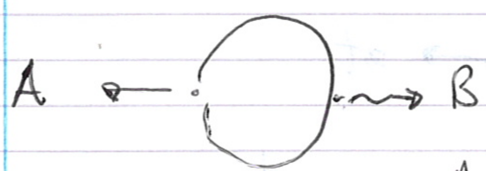
$\chi(K(n,k)) \leq n-2k+2$ also $\chi \geq \frac{\#\text{ vertices}}{\text{indep }\#} = n/k$

Also $\chi \geq$ clique $= \left\lfloor \frac{n}{k} \right\rfloor$

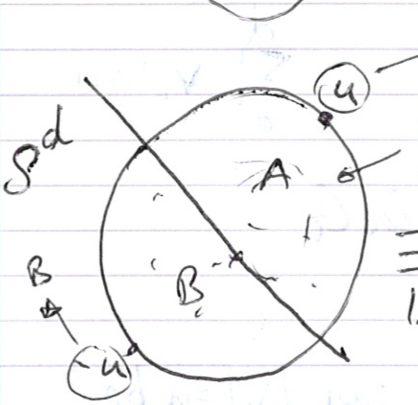
$\exists$ homomorphism $\varphi : H \to G \Rightarrow \chi(G) \geq \chi(H)$

We will prove that $\chi(K(n,k)) \geq n-2k+2$ by finding a hom$^{\underline{m}}$ Borsuk$(d, \varepsilon) \longrightarrow$ Kneser(n,k) where $d = n-2k$, $\varepsilon > 0$ small.

(Borsuk(d,ε), $V = S^d$ for $x, y \in S^d$ $\quad xy \in E \Leftrightarrow \|x+y\| < \varepsilon$.)



$A \cap B = \emptyset$.



n points which have the property in every (open) hemisphere of S^d there are k points.

$\exists A \subseteq n$-point set $\cap$ hemisphere s.t. $|A| \geq k$.

So u (a point in Borsuk) $\longrightarrow A$ a set of $\geq k$ points in $[n]$.

For $d=1$, $n = 2k+1$ $\qquad S^d = S^1$.

<u>Moment curve</u>: $\underset{\mathbb{R}}{t} \to (1, t, t^2, \dots, t^d) = \mu(t)$.

<u>Goal</u>: Find n pts in $\mathbb{R}^{d+1} / \{0\}$ s.t. every half-space that is determined by a ~~hypertane~~ hyperplane through the origin contains $\geq k$ of the pts.

$\{p_j : j = 1, \dots, n\}$ with $p_j = (-1)^j \mu(j)$.

$\vec{a} = (a_0, \dots, a_d) \Rightarrow$ half space $\{x \in \mathbb{R}^{d+1} : \sum_{i=0}^{d} a_i x_i > 0\} = H^+$

Is $\mu(t) \in H^+$? Well $a_0 + a_1 t + \dots + a_d t^d > 0$ if $\mu(t) \in H^+$

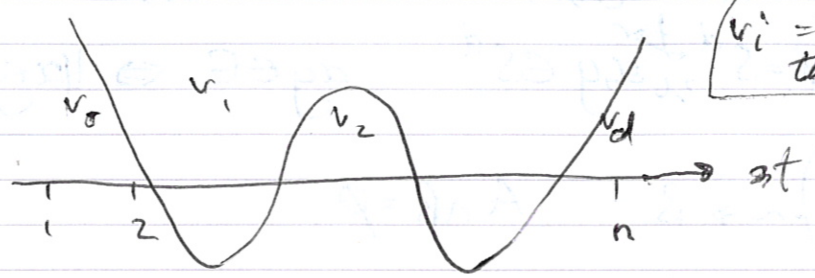
Defⁿ: A as the polynomial $u \to \sum_{j=0}^{d} a_j u^j$ ~~$\mu(t)$~~

So $\mu(t) \in H^+ \Leftrightarrow A(t) > 0$

CLAIM: $\forall$ polynomial A of degree $\leq d$ at least k (and at most $k+d$) of the ~~inequalities~~ ~~conditions~~ inequalities

~~sgn~~ $(-1)^j (A(j)) > 0 \quad j = 1, \dots, n$ hold.

v_i = the no. of integer points that $A(t)$ hits ?



$\left\lfloor \frac{v_i}{2} \right\rfloor \leq$ Successes in the ith interval $\leq \left\lceil \frac{v_i}{2} \right\rceil \qquad \sum_{0}^{d} v_i = n$

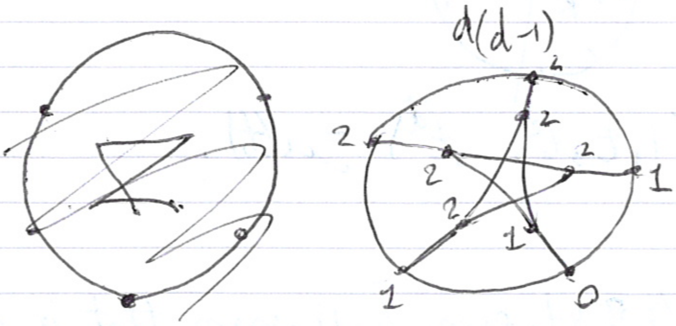
girth(G) = least length of a cycle in G.

Sparsity: Want girth to be large.

$g = 5$, G d-regular how small can $|V| = n$ be?

Clearly, $n \geq 1 + d + d(d-1)$

$n \geq d^2 + 1$



Question: How many graphs are there s.t. they are d-regular $n = d^2 + 1$ and girth $= 5$.

Thm: There are either four or ~~five~~ three values of d for which $\exists G$ which is d-regular, $n = d^2+1$, girth $=5$. $d = 57$?

$d=2$ 5-cycle $n=5$
$d=3$ Petersen Graph $n=10$
$d=7$ Hoffman–Singleton $n=50$
$d=57$ Unknown $n=3250$

$$G \longrightarrow \begin{array}{cc} & j \\ i & \begin{pmatrix} 0 & & \\ & 1 & \\ & & 0 \end{pmatrix} \end{array} = A \qquad A\vec{1} = d\cdot\vec{1}$$

1 if $ij \in E(G)$

e-vector, e-value.

Adjacency Matrix

A^2_{ij} $= \#$ ways of going from i to j in precisely two steps

$$A^2 = dI + \underbrace{J - A - I}_{\text{non-neighbour matrix}} \qquad J = \begin{pmatrix} 1 & \cdots & 1 \\ \vdots & & \vdots \\ 1 & \cdots & 1 \end{pmatrix}.$$

(because $g=5$ and $n = d^2+1$)

If $\vec{u}$ is an eigenvector with e-value $\lambda \neq d \Rightarrow \langle u, \vec{1} \rangle = 0$

$$A^2\vec{u} = dI\vec{u} + J\vec{u} - A\vec{u} - \vec{u}$$
$$\lambda^2 = d + 0 - \lambda - 1$$

i.e. $\lambda^2 + \lambda + 1 - d = 0$

$$\lambda_{1,2} = \frac{-1 \pm \sqrt{\cancel{(d-1)^2 - 4}\; 1^2 + 4(d-1)}}{2}$$
$$= \frac{-1 \pm \sqrt{4d-3}}{2}$$

$m_1 + m_2 = n - 1 = d^2$ counting multiplicity.

$m_1\lambda_1 + m_2\lambda_2 + d = \operatorname{tr} A = 0$