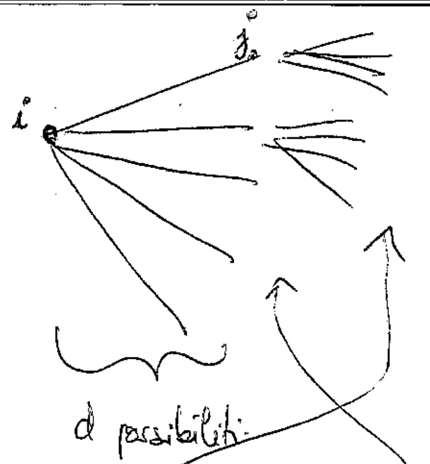


Se o j estiver "ir e vir" não é possível.

se o j estiver há 1 possibilidade de ir de i p/ j (ir e vir)

$$A = \begin{array}{cc} & \begin{array}{cc} j & j' \end{array} \\ i & \begin{pmatrix} 0 & & 1 & 0 \\ & \ddots & & \\ & & \ddots & \\ & & & 0 \end{pmatrix} \end{array}$$

$$A^2 = \begin{pmatrix} d & 0 & 1 \\ & & \\ & & \end{pmatrix}$$

$$A + A^2 = \begin{pmatrix} d & & 1 \\ & \ddots & \\ 1 & & d \end{pmatrix}$$

2ª Parte

última sessão.

AFTERNOON SESSION – LINIAL
26 JULY 2012

Statement of the Thm

Let G be a connected d-regular graph on $n = d^2+1$ vertices with girth $= 5$

$$\Downarrow$$

$$d = \{2, 3, 7, 57\}$$

Eigenvalues of G's adjacency matrix were:

$$d, \quad \lambda_1 = \frac{-1+\sqrt{\Delta}}{2}, \quad \lambda_2 = \frac{-1-\sqrt{\Delta}}{2}$$

$\lambda_1 \downarrow m_1$ (multiplicity), $\lambda_2 \downarrow m_2$, $\quad \Delta = 4d-3$

- $m_1 + m_2 = n-1 = d^2$
- $d + m_1\lambda_1 + m_2\lambda_2 = 0$ (trace condition)

$$\Leftrightarrow d + m_1\left(\frac{-1+\sqrt{\Delta}}{2}\right) + m_2\left(\frac{-1-\sqrt{\Delta}}{2}\right) = 0$$

$$\Leftrightarrow d - \frac{m_1+m_2}{2} + \frac{\sqrt{\Delta}}{2}(m_1 - m_2) = 0$$

$$\Leftrightarrow d - \frac{d^2}{2} + \frac{\sqrt{\Delta}}{2}(m_1 - m_2) = 0$$

$$\begin{cases} \Leftrightarrow \ \Delta \text{ must be a square number} \\ \Leftrightarrow \ \Delta = t^2 = 4d - 3 \\ \Leftrightarrow \ d = \dfrac{t^2+3}{4} \end{cases}$$

then
$$0 = \underbrace{\frac{t^2+3}{4} - \frac{(t^2+3)^2}{32} + \frac{t}{2}(m_1 - m_2)}_{} = 0$$

the constant term is 15

so we have

$$15 + t\left(\qquad\qquad \right) = 0$$

so $t \mid 15$ $\quad t \in \{1, 3, 5, 15\}$

Question:

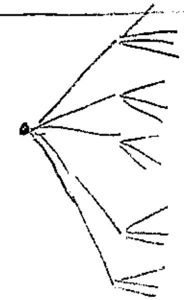
How large can the girth be for a d-regular graph on n vertices?

$g(d,n) \to$ the large girth

Theorem

$$\left(2 + o(1)\right)\frac{\log n}{\log(d-1)} \geq g(d,n) \geq \left(\frac{4}{3} - o(1)\right)\frac{\log n}{\log(d-1)}$$

(*)



$$d \quad d(d-1) \quad d(d-1)^2 \quad \dots \quad d(d-1)^{g/2} \leq n$$

Conjecture of LINIAL

we can substitute in (*) $2 + o(1)$ by

$$(2 - \varepsilon_0), \quad \varepsilon_0 > 0 \text{ for some } \varepsilon_0.$$

In fact (*) is due to Lubotzky & Philips Sarnak

First we will show that the lower bound of (*), where $\frac{4}{3} - o(1)$ is replaced by 1, an achievement of Erdős and Sachs:

Theorem (Erdős, Sachs)

$\exists$ d-reg graphs with n vertices and

$$\text{girth} \geq \left(1 - o(1)\right)\frac{\log n}{\log(d-1)}$$

Step 1

Construct a d-reg graph with the desired girth and ($N >> n$ vertices)

[Note
$$\operatorname{diam}(G) \geq \left\lceil \frac{\text{girth}(G)}{2} \right\rceil$$
]

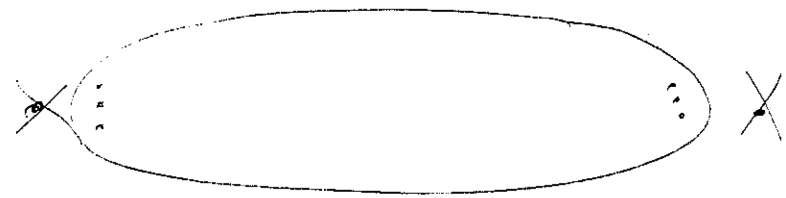
If G is d-reg has n vertices $\Rightarrow$
$$\operatorname{diam}(G) \geq (1 + o(1)) \frac{\log n}{\log(d-1)}$$

$\to$ graph with too many vertices.

We will maintain g (girth) and keep reducing the n° of vertices as long as $g < \operatorname{diam}(G)$. [I will eliminate vertices without reducing the girth]

But when $g \geq \operatorname{diam}(G)$ and as
$$\operatorname{diam} \geq \frac{\log n}{\log(d-1)}.$$
Again: As long as the girth < diameter (G) we want to find a way for eliminating vertices without reducing the girth. How to do this?

Pick two vertices which are at distance = diameter (Remember: maintain the regularity)



then we connect the others in order to maintain regularity

But haven't we changed the girth? Is there a possibility that we've created a too short cycle. Now the dist $\geq$ diam -2

So the length of newly created cycles $\geq$
$$1 + \operatorname{diam} - 2 \geq g$$

CODES ...

A BINARY CODE. $C \subseteq \{0,1\}^n$

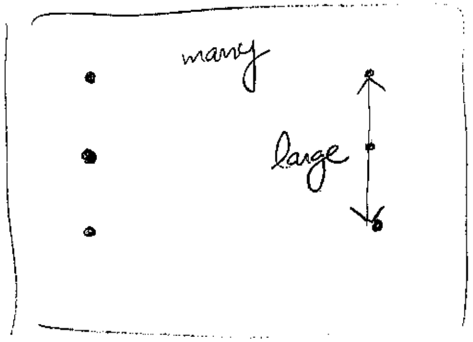
$v \in C$ is called word

we want

$|C|$ big, and $\operatorname{distance}(C) = \min_{x \neq y \in C} d_H(x,y)$

$$\forall\, x, y \in C \Rightarrow d_H(x,y) = \left| \{ i \mid x_i \neq y_i \} \right|$$

n-dimensional cube



Linear Codes

Pick a 0/1 matrix $A_{m \times n}$

and

$$C = \{ x \in \mathbb{F}_2^n \mid Ax = 0 \}$$

We can assume that the rows of A are l. independent.

$$\text{Rank}(A) = m$$

$$|C| = 2^{n-m}$$

$$\text{dist}(C) = \min \{ \underbrace{|y|}_{\text{weight}} \mid y \neq 0, y \in C \}$$

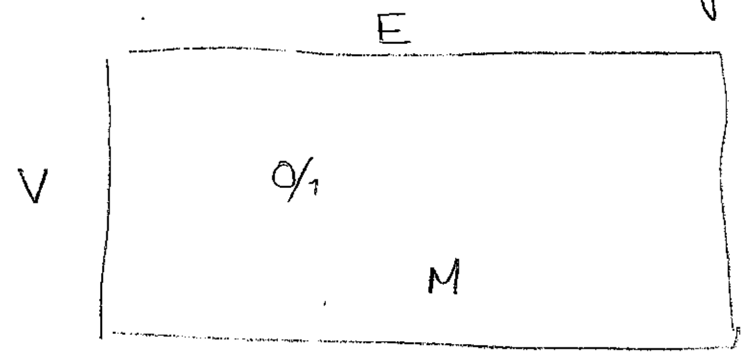
dist(C) := least number of col's in A whose sum is 0

(= shortest linear dependence among the col's of A)

$\boxed{\text{BOP}}$ Big Open Problem

Given $1 > \alpha > 0$, what is the largest β s.that $\exists_{\underbrace{\alpha n}_{m} \times n}$ 0/1 matrices s.t. no set of $< \beta n$ col's in linearly dependent?

How is this connected with the girth of a graph?



$\{ x \mid Mx = 0 \}$ incidence matrix of $G = (V, E)$