Lectures on Representation Theory of Finite Groups, Problem Sheet 1

Throughout, G is a finite group with identity element e. Recall C_n is the cylic group with n elements; S_n is the symmetric group of permutations on n letters; D_n is the group of all rotation and reflection symmetries of the regular n-gon.

The cardinality of a set X is denoted $|X|$.

1. Consider the representations $\pi_k : C_n \to GL_2(\mathbb{R})$ of C_n defined by

$$\pi_k(x^j) = \begin{pmatrix} \cos(\frac{2\pi jk}{n}) & \sin(\frac{2\pi jk}{n}) \\ -\sin(\frac{2\pi jk}{n}) & \cos(\frac{2\pi jk}{n}) \end{pmatrix}$$

where $k \in \{0, 1, 2, \ldots, n-1\}$ and x is a generator of C_n. Notice that the matrix above is a clockwise rotation by $\frac{2\pi jk}{n}$ radians.

   (a) Show that over $\mathbb{C}$, π_k is reducible for each k by finding a one-dimensional subrepresentation. Indeed, π_k is the direct sum of two one-dimensional subrepresentations. (Hint: it is enough to focus on the case $j = 1$.)

   (b) Show that for each k, π_k is irreducible as a representation over $\mathbb{R}$, except when $k = 0$ or $k = \frac{n}{2}$ when n is even.

   (c) Show that π_k and π_{n-k} are equivalent as representations over $\mathbb{R}$ (and hence also $\mathbb{C}$) by finding a real 2×2 matrix Q such that

$$Q\pi_k Q^{-1} = \pi_{n-k}$$

   as homomorphisms from C_n to $GL_2(\mathbb{R})$.

2. Let $\mathbb{H}$ be the quaternion algebra over the rational numbers. So $\mathbb{H}$ is a 4-dimensional vector space over $\mathbb{Q}$ with basis $1, i, j, k$ and the multiplication is given by $i^2 = j^2 = k^2 = -1$, $ij = k = -ji$, $jk = i = -kj$, and $ki = j = -ki$. Furthermore, the line spanned by 1 defines a copy of the field $\mathbb{Q}$ in $\mathbb{H}$ which is exactly the center of $\mathbb{H}$.

   Let Q_8 be the group of 8 elements, $\pm 1, \pm i, \pm j, \pm k$, with multiplication defined from $\mathbb{H}$. This is called the quaternion group.

   Notice that the action of an element $x \in Q_8$ by left multiplication on $\mathbb{H}$ can be viewed as a $\mathbb{Q}$-linear transformation of the underlying 4-dimensional rational vector space. Furthermore, left multiplication on $\mathbb{H}$ defines a 4-dimensional representation π of Q_8 over $\mathbb{Q}$.

   (a) Write down the matrix for $\pi(i)$ relative to the standard basis $\{1, i, j, k\}$ of $\mathbb{H}$.

   (b) Show that $1 \pm \sqrt{-1}\, i$ and $j \pm \sqrt{-1}\, k$ are eigenvectors for $\pi(i)$ over $\mathbb{C}$.

   (c) Show that $1 - \sqrt{-1}\, i$ and $j + \sqrt{-1}\, k$ span a subrepresentation of π, where we are now viewing π as a complex representation. Write down the 2×2 complex matrices for each element of Q_8 relative to this basis of the subrepresentation.

3. Let F be an algebraically closed field (but no restriction on the characteristic). Show that every irreducible representation defined over F of an abelian group is one-dimensional.

   Hint: Let V be an irreducible representation of G. First, pick some $g \in G$. Since F is algebraically closed, g has a nonzero eigenspace on V, say of eigenvalue λ. So $E_\lambda = \{v \in V \,|\, g.v = \lambda v\}$ is nonzero. Show that E_λ is preserved by all $g' \in G$. In other words, E_λ is a subrepresentation of V. Conclude that $V = E_\lambda$ and further that $V = E_\lambda$ cannot be irreducible unless it is one-dimensional.

4. (**I suggest reading this exercise for background, but not spending time on its details, which are fairly straightforward, but not especially enlightening**)

   Recall the definition of a group action. A **group action** for a group G acting on a set X refers to a map
   $$\psi : G \times X \to X,$$
   usually written $\psi(g, x) = g.x$, satisfying two axioms: $e.x = x$ and $(gh).x = g.(h.x)$. Here, of course, $x \in X$ and $g, h \in G$.

   Let $\mathrm{Aut}(X)$ be the group of invertible maps from X to itself, with multiplication given by composition of maps. Show that a group action defines a homomorphism of groups
   $$\pi : G \to \mathrm{Aut}(X)$$
   and conversely, that every such homomorphism defines a group action. Furthermore, if $|X| = n$, and we identify X with $\{1, 2, \ldots, n\}$, then an action of G on X is just a homomorphism of groups $\pi : G \to S_n$.

   Notice that these results are analogous to our definition of a group representation: instead of $\mathrm{Aut}(X)$ and S_n for a group action, the target groups are replaced by $GL(V)$ and $GL_n(F)$, where V is a vector space over the field F.

5. Every group action of G on a set X gives rise to a representation of G. Namely, let V_X be a vector space over a field F with a basis
   $$\{e_x\}_{x \in X}.$$
   In other words, the set X indexes a basis of V_X. Then the corresponding group representation is defined by
   $$g.(e_x) := e_{g.x}$$
   and extending linearly. It is called the **permutation representation** of G coming from the action on X.

   Let C_3 act on itself by left multiplication, i.e., $X = C_3$. With $F = \mathbb{C}$, write V_X as the direct sum of three one-dimensional (hence, irreducible) representations. With $F = \mathbb{Q}$, show that V_X is the direct sum of a one-dimensional representation and a two-dimensional irreducible representation. With $F = \mathbb{F}_3$ (the finite field of three elements), V_X cannot be written as the direct sum of irreducible representations.

   Incidentally, the permutation representation arising from G acting on itself by left multiplicaiton is called the **regular representation** of G.

   Notice that this exercise and also exercise 1 show that the restriction on F being algebraically closed is necessary for the result in exercise 3. This exercise also shows that the restriction on characteristic is necessary for Maschke's Theorem on complete reducibility.

6. Let G act on the finite set X. A few facts to recall about group actions: the set X will decompose under G into orbits $\mathcal{O}_1, \ldots, \mathcal{O}_m$. If we consider the action of G on any orbit $\mathcal{O}_i$, the action is transitive, and we can identify the action with the action of G on the set of left cosets G/H_x, where H_x is the stabilizer of an element $x \in \mathcal{O}_i$. Recall, $H_x = \{h \in G \mid h.x = x\}$. Another useful fact is that $x, y \in \mathcal{O}_i$, with $y = g.x$, then $H_y = gH_xg^{-1}$.

   For $g \in G$, define $\mathrm{Fix}(g) = \{x \in X \mid g.x = x\}$.

   (a) Show that
   $$\frac{1}{|G|} \sum_{g \in G} |\mathrm{Fix}(g)| = m,$$
   the number of orbits of G on X. This is due to Frobenius, but is often called Burnside's Lemma. Hint: First, it is enough to prove this for a transitive action (why?). That is, we may assume $X = \mathcal{O}_1$. Now show that $\sum_{g \in G} |\mathrm{Fix}(g)| = |G|$ by instead showing that $\sum_{x \in X} |H_x| = |G|$.

(b) Let K be the group of rotational symmetries of a cube. That is, K is set of all rotations (about the center of the cube) which take the cube to itself. Show that K has 24 elements: the identity; 3 rotations of order 2 about an axis through the center a face; 6 rotations of order 4 about an axis through the center a face; 8 rotations of order 3 about an axis through a vertex; and 6 rotations of order 2 about an axis through the center an edge. (You might also show that K is isomorphic to S_4).

(c) You have q colors to paint the faces of the cube. Each face can be any of the q colors. Two of the possible q^6 coloring are considered equivalent if they are indistinguishable as pieces of artwork (that is, up to the action of the group K from the previous part).

Use Burnside's Lemma to count the number of equivalent colorings of the cube. A variant of this problem was part of a written quiz Google asked prospective applicants to complete (circa 2005).