Lectures on Representation Theory of Finite Groups, Problem Sheet 2

Throughout, G is a finite group with identity element e. Recall C_n is the cylic group with n elements; S_n is the symmetric group of permutations on n letters; D_n is the group of all rotation and reflection symmetries of the regular n-gon.

The cardinality of a set X is denoted $|X|$.

1. Let G act on the finite set X. Let (V, π) be the corresponding permutation representation (see Problem Sheet 1). Define

$$V^G = \{v \in V \mid g.v = v\}.$$

This is called the invariant space of G on V. It is clearly a subrepresentation of V and it is a direct sum of trivial representations of G.

   (a) Show that the dimension of V^G equals the number of orbits of G on X.

   (b) For $g \in G$ note that $\pi(g)$ is a permutation matrix relative to the basis indexed by X. Conclude that the trace of $\pi(g)$ is equal to $|\mathrm{Fix}(g)|$.

   (c) Use Exercise 6 on Problem Sheet 1 to conclude that

$$\dim(V^G) = \frac{1}{|G|}\sum_{g \in G} \mathrm{tr}(\pi(g)).$$

   We will generalize this result to an arbitrary representation of G in the coming lectures.

2. (Classification of irreducible representations of a finite abelian group G over $F = \mathbb{C}$). Let G be a finite abelian group. From Problem Sheet 1, we know that every irreducible representation of G is one-dimensional. From Lecture 1, we know how to classify the one-dimensional representations of the cyclic group C_n. They are given by assigning a generator $x \in C_n$ to any of the n-th roots of unity in $\mathbb{C}$. Let us write $\mathrm{Hom}(G, A)$ for the set of group homomorphisms of G to an abelian group A. When A is the multiplicative group $F^\times$ of a field, $\mathrm{Hom}(G, F^\times)$ is called the character group (or characters) of G and is denoted $\widehat{G}$, for any group G (not just abelian).

   (a) Show that $\mathrm{Hom}(G, A)$ is itself an abelian group by defining $f_1 \star f_2$ by

$$f_1 \star f_2(g) := f_1(g) f_2(g).$$

   (So the only thing to show is that $f_1 \star f_2$ is again a homomorphism from G to A.)

   (b) Convince yourself that the one-dimensional representations of any G are nothing but the set $\mathrm{Hom}(G, \mathbb{C}^*)$. So the one-dimensional representations have the additional structure of an abelian group. Convince yourself that

$$\mathrm{Hom}(C_n, \mathbb{C}^*) \cong \mathrm{Hom}(C_n, C_n) \cong C_n.$$

   These isomorphisms are not canonical since they require choosing generators of the various groups, which are not unique.

   (c) Show that $\mathrm{Hom}(G \times H, A) \cong \mathrm{Hom}(G, A) \times \mathrm{Hom}(H, A)$ for two groups G, H.

   (d) Since every finite abelian group is a direct product of cyclic groups, use the previous results to show that $\widehat{G} \cong G$ for G a finite abelian group. The isomorphism is not canonical, but it does give us a way to parametrize the irreducible representations of G.

   (e) Write down the character table for the abelian groups C_4 and $C_2 \times C_2$.

3. (Classification of one-dimensional representations of an arbitrary finite group G over $F = \mathbb{C}$).

(a) (General useful fact) Let H be a normal subgroup of G. Let $p : G \to G/H$ be the quotient homomorphism.

Show that any representation of the quotient group G/H defines a representation of G (hint: compose with p).

Conversely, show that any representation π of G with H in the kernel of π defines a representation of G/H. Conclude that there is a bijective correspondence between representations of G/H and representations of G with H in the kernel of the representation. (This is an explicit use of the basic properties of quotient groups, embedded in the isomorphism theorems for quotient groups).

(b) Let $\pi : G \to \mathbb{C}^*$ be a one-dimensional representation of G (that is, a *character* of G). Show that the commutator subgroup of G lies in the kernel of π. Recall the commutator subgroup of G denoted $[G, G]$ is the subgroup of G generated by the set of pure commutators

$$\{xyx^{-1}y^{-1} \mid x, y \in G\}.$$

Conclude that the one-dimensional representations of G coincide with those of $G/[G, G]$.

(c) More is true. Show that $G/[G, G]$ is abelian. Therefore the one-dimensional representations of G are equal in cardinality to $|G/[G, G]|$ and are easily found once we know $G/[G, G]$.

You might also prove: $[G, G]$ is contained in any subgroup H of G where G/H is abelian.

(d) Let $G = D_4$. Denote by $Z(G)$ the center of a group. Show that $[G, G] = Z(G)$ is of order two and conclude that $G/[G, G]$ is isomorphic to $C_2 \times C_2$. Use this to find the one-dimensional representations of D_4. (Facts from group theory: groups of order 4 are abelian and $G/Z(G)$ cannot be cyclic if G is nonabelian). Do the same for $G = Q_8$ (see Problem Sheet 1 for the definition).

(e) Let $G = S_n$. Recall that S_n is generated by the simple transpositions $(ii + 1)$ and that the simple transpositions lie in the same conjugacy class. Use this to show that S_n has at most two 1-dimensional representations. Use the determinant to show that S_n does in fact have a non-trivial representation (called the sign representation). What does this say about the relationship between the commutator subgroup of S_n and the index two subgroup A_n (the alternating group)?

Some problems for after Lecture 4 (some of these may be done in lecture):

1. Find the character table for D_4 and Q_8.

2. Find the character table for A_4.

3. Recall that S_n has a natural permutation representation on $V = \mathbb{C}^n$, with character χ_V, which decomposes into a copy of the trivial representation spanned by $\sum e_i$ and another representation V_{std} given by

$$\{v = \sum a_i e_i \in V \mid \sum a_i = 0\}.$$

This latter is called the standard or defining representation of S_n. Here are two proofs that V_{std} is irreducible:

(a) Pick any $v \in V_{std}$ nonzero and show that the set of vectors $\{\sigma.v \mid \sigma \in S_n\}$ span V_{std}. (Hint: first, show that the set contains a vector with $a_i \neq a_{i+1}$ for any i. Second, show that $e_i - e_{i+1}$ lies in the span of the set by using the simple transposition $(ii + 1)$.)

(b) The action of S_n has one orbit on $X = \{1, 2, \ldots, n\}$. Show that it has two orbits on $X \times X$. Next, show that the permutation representation of S_n on $X \times X$ has character $(\chi_V)^2$ (see Problem 1(b) above). Since the character χ_V is real, we may also interprete $(\chi_V)^2$ as $\langle \chi_V, \chi_V \rangle$, which is the number of irreducible representations in V. Conclude, from Problem 1, that V has two (distinct) irreducible constituents.

4. Find the character table for S_5 and S_6. (Hint: recall that $\mathrm{Hom}(V, W)$ is a representation and its character is $\chi_V \bar{\chi}_W$.)