Representations of finite groups

- Traced back to Frobenius in 1896-7, who invented characters of finite groups, after which his stude Burnside took it up.
- G will always be a finite group, with identity e
- $\forall g \in G$ assign an $n \times n$ matrix over some field $\mathbb{F}$.
- Required that $e \rightsquigarrow \begin{pmatrix} 1 & & 0 \\ & 1 & \\ 0 & & 1 \end{pmatrix}$ and for $g \rightsquigarrow A_g$ and $h \rightsquigarrow A_h$ we have $g \cdot h \rightsquigarrow A_g \cdot A_h$
- We study all possible such representations.
- $\forall g \in G$: A_g has to be invertible.

E.g.
- $D_4 = \langle x, y \mid x^4 = y^2 = e, y y^{-1} = x^{-1} \rangle$ dihedral group, acti on the square.
- $x \rightsquigarrow \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ $90°$ clockwise rotation
- $y \rightsquigarrow \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

E.g.
- $C_3 = \langle x \mid x^3 = e \rangle$
- Only 3 1-dimensional representations: $\{1, e^{\pi i/3}, e^{4\pi i/3}\}$.

Def.
- Let V be finite-dimensional vector space over field $\mathbb{F}$.
- $GL(V) = \{\theta : V \to V \mid \theta \text{ linear, invertible}\}$ general linear group, group under composition.
- $End(V) = \{\theta : V \to V \mid \theta \text{ linear}\}$ endomorphisms.
- If we choose basis $e_1, \dots, e_n$ of V, then $GL(V) \cong GL_n(\mathbb{F})$ ← $n \times n$ invertible matrices over $\mathbb{F}$.

Def.

- A (linear) representation of G is a pair (V, π), wh V is a vector space over F and $\pi : G \to GL(V)$ is a homomorphism of groups $(\pi : G \to GL_n(F))$.

Def.

- The dimension (degree) of π is $\dim_F(V)$
- $\forall_{g \in G, v \in V}: g.v = \pi(g)v$.
- $$\left.\begin{array}{l} g.(v+w) = g.v + g.w \\ g.(\lambda v) = \lambda(g.v) \quad \forall \lambda \in F \\ e.v = v \quad \forall v \in V \\ (gh).v = g.(h.v) \end{array}\right\} \text{group action axioms}$$

E.g

- $w = e^{2\pi i/3}$, $\mathbb{Q}(w) = \{\mathbb{Q} \cdot 1 + \mathbb{Q} \cdot w\}$
- $\dim_{\mathbb{Q}} \mathbb{Q}(w) = 2$
- $w^2 = -1 - w$
- $(-w)^6 = 1$
- $C_6 = \langle x \mid x^6 = 1 \rangle$
- $x \rightsquigarrow -wx$
- $e_1 = \{1\}$, $e_2 = \{w\}$
- $w \rightsquigarrow \begin{pmatrix} 0 & 1 \\ -1 & 1 \end{pmatrix}$ ← order 6 matrix

Remark

- $g \in G$ with order $m \Rightarrow \underset{\pi(g)^m}{\overset{\parallel}{\pi(g^m)}} = \pi(e) = id_V$

E.g

- For C_n take $x \rightsquigarrow n^{th}$ root of unity in $\mathbb{C}^\times$
- $\pi_k(x) = \left(e^{\frac{2\pi i}{n}}\right)^k$

Exercise

- If G is an Abelian group and F is algebraically clos then every irreducible representation is one-dimension

Def.

- A subrepresentation of (V, π) is a subspace $W \leq V$ such that $\pi(g)(W) \subseteq W$ for all $g \in G$.

Def.

- (V, π) is irreducible if 0 and V are the only subrepresentation.

Goal

- Try to understand/classify the irreducible representations of G.
- If G is cyclic, there are $n = |G|$ 1-dimensional irreducible representations.

E.g.

- Let S_n = symmetric group with $n!$ elements. S_n acts on $X = \{1, 2, \ldots, n\}$.
- Let $V_X = \langle e_1, e_2, \ldots, e_n \rangle$ be the vector space gen. by $e_1, \ldots, e_n$.
- If G acts on set X, then V_X w/ basis $(e_x)_{x \in X}$ and $g.e_x = e_{g.x}$ is linear rep.
- $(1\ 2\ 3) \in S_3 \rightarrow \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$
- $W := \langle e_1 + e_2 + e_3 \rangle \leq V$ subrep

  $W' := \langle e_1 - e_2, e_2 - e_3 \rangle \leq V$ sub rep.
- $V = W \oplus W'$

Thm.

- Given W, W' two reps. of G. Then $W \oplus W'$ is a rep. of G, with $g.(w,w') := (g.w, g.w')$ for $(w,w') \in W \oplus W'$.

Thm (Maschke's theorem)

- If F has characteristic 0 or char $F \nmid |G|$ the every rep V of G decomposes into

$$V = W_1 \oplus W_2 \oplus \cdots \oplus W_n,$$

where w_i are irreducible.

Def.

- Let A, B be matrices. Then $A \sim B$ iff $\exists Q \in GL_n($ invertible with $QAQ^{-1} = B$.

- $g \in G \longrightarrow A_g \in GL_n(F)$

  $\longrightarrow B_g \in GL_n(F)$

  Then the two reps. iff there is one matrix for all A_g:

$$Q A_g Q^{-1} = B_g$$

- $(V, \pi) \sim (W, \pi')$ means there is $T: V \to W$ invertib such that

$$\begin{tikzcd} V \arrow[r, "\pi(g)"] \arrow[d, "T"'] & V \arrow[d, "T"] \\ W \arrow[r, "\pi'(g)"'] & W \end{tikzcd}$$

is commutative, i.e $\forall g \in G: T \circ \pi(g) = \pi'(g) \circ T$ and $T \circ \pi \circ T^{-1} = \pi'$.