Schurs Lemma

1) V, W irreducible and $V \not\cong W$
$\Rightarrow \mathrm{Hom}_G(V,W) = \{0\}$ (any F)

2) V irred $\Rightarrow \mathrm{End}_G(V) := \mathrm{Hom}_G(V,V)$ is a ring under composition (in fact an associative F-algebra) and in fact a division ring (skew field)

An example of a ~~fie~~ division ring which is not a field is Hamilton's quaternion ring $\mathbb{H}$.

3) If V irred, $F = \mathbb{C}$, then

$$\mathrm{Hom}_G(V,V) \simeq \mathbb{C}.$$

These maps are exactly scalar multiples of identity.

Proof: 1) $\varphi : V \longrightarrow W$
$\ker \varphi \subset V$ subrep
$\mathrm{im}\, \varphi \subset W$ subrep

Indeed let $v \in \ker \varphi$ and let $g \in G$.
$\varphi(g.v) = g.\varphi(v) = 0$ so $g.v \in \ker \varphi$
Let $w \in \mathrm{im}\,\varphi$. $\exists v \in V : \varphi(v) = w$.
Now $g.w = \varphi(g.v) \in \mathrm{im}\,\varphi$.

If $\varphi \neq 0$, then it has trivial kernel (so ~~is~~ φ is injective) and its image is W (so φ is surjective) and hence φ is an isomorphism.

2) Any non-zero morphism is invertible so that $End_G(V)$ is a division ring.

3) Let $\alpha \in End_G(V)$. $\underline{\alpha \text{ has an eigenvector}}$ (say with eigenvalue λ) i.e.

$$\ker[\alpha - \lambda I] \neq 0.$$

since $\mathbb{C}$ is algebraic closed.

But $\ker[\alpha - \lambda I]$ is a non-zero subrepresentation of V so that $\ker[\alpha - \lambda I] = V$ and

$$\alpha - \lambda I \equiv 0.$$

Now

$$\alpha = \lambda I.$$

$$\text{Now } End_G(V) = \{\lambda I : \lambda \in \mathbb{C}\} \cong \mathbb{C}.$$

Example: $\mathbb{Q}(\omega)$ $\omega = \sqrt[3]{-1}$

$$\begin{array}{c} \mathbb{Q}(\omega) \\ | \\ \mathbb{Q} \end{array}$$

Representation of C_6 on $\mathbb{Q}(\omega)$ given by left mul of $-\omega$. ~~[F = Q representation]~~

This 2-dim representation is irreducible over $\mathbb{Q}$.

Now by Schur, $End_G(\mathbb{Q}(\omega))$ is a division algebra.

$\cancel{End_G(\mathbb{Q}(\omega)}$

$$End_G(V \cong \mathbb{Q}^2) \subset End(\mathbb{Q}^2) \cong M_2(\mathbb{Q})$$

$$\mathbb{Q} = \left\{ \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix} \right\} \subset \mathbb{Q}(\omega) \subsetneq M_2(\mathbb{Q})$$

$$\mathrm{End}_G(V) \cong \mathbb{Q}(\omega)$$

$$\Downarrow \quad \text{field}$$

$$\left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ -1 & 1 \end{pmatrix} \right\}$$

Maschke's Thm:

$$V = (\underbrace{V_1 \oplus \cdots \oplus V_1}_{a_1}) \oplus (\underbrace{V_2 \oplus \cdots \oplus V_2}_{a_2}) \oplus \cdots \oplus (\underbrace{V_k \oplus \cdots \oplus V_k}_{a_k})$$

V_i are irred subreps mutually non-isomorphic

W an arbitrary irreducible rep.

$$\mathrm{Hom}_G(W, V) = \mathrm{Hom}_G(W, \bigoplus_{i=1}^{k} a_i V_i)$$

$$\cong \bigoplus_{i=1}^{k} a_i \, \mathrm{Hom}_G(W, V_i)$$

as vector spaces over $\mathbb{C}$.

Everything dissappears except for the case of $V_j \cong W$.

So $\mathrm{Hom}_G(W, V) \overset{\text{v.sp}}{\cong} \begin{cases} \oplus a_j \, \mathrm{Hom}_G(W, V_j) & \text{if } W \cong V_j \\ 0 & \text{if no such } j. \end{cases}$

So $\dim_{\mathbb{C}} \mathrm{Hom}_G(W, V) = a_j$ if $W \cong V_j$ appears in the decomposition of V. ~~[illegible]~~ This result counts the multiplicity of W in the decomposition.

$$V = (\underbrace{V_1 \oplus \cdots \oplus V_1}_{W_1}) \oplus \cdots \oplus (\underbrace{V_j \oplus \cdots \oplus V_j}_{W_j})$$

Claim: $W_i = \sum U$ where $U \subset V$ subrep and $U \cong V_i$.

i.e. W_i is uniquely determined.

"V_i - isotypic components of V".

Characters

$\chi : G \longrightarrow \mathbb{C}^\times$ is called a character.

$\hat{G} := \mathrm{Hom}(G, \mathbb{C}^\times)$ group homs

Abelian gp

If (V, π) is an arbitrary rep. then we define the linear character of V denoted $\chi_V : G \longrightarrow \mathbb{C}$ is by definition

$$\chi_V(g) = tr(\pi(g)) = \sum_{i=1}^{n} \lambda_i.$$

Facts: (1) $V \cong W \Rightarrow \chi_V = \chi_W$

(2) If g, h are conjugate in G and π is a gp morphism with domain G, then $\pi(g)$ and $\pi(h)$ are conjugate, so $tr\, \pi(g) = tr\, \pi(h)$.

$\Rightarrow \chi_V(g) = \chi_V(h)$

ie χ_V constant on conjugacy classes "class functions".

(3) V is 1-dimensional: χ_V ~~agrees with~~ agrees with existing definition.

(4) $\chi_V(g) = \sum \lambda_i$ where $\pi(g) \sim \begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{pmatrix}$.

since λ_i are roots of unity

$$\begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{pmatrix}^{-1} = \begin{pmatrix} \lambda_1^{-1} & & \\ & \ddots & \\ & & \lambda_n^{-1} \end{pmatrix} = \begin{pmatrix} \bar{\lambda}_1 & & \\ & \ddots & \\ & & \bar{\lambda}_n \end{pmatrix}$$

$$tr\, \pi(g^{-1}) = \sum \bar{\lambda}_i = \overline{\sum \lambda_i} = \overline{tr\, \pi(g)}$$

$$\chi_V(g^{-1}) = \overline{\chi_V(g)}$$

Character Table for G

$C_3 = G$

	1	x	x^2
triv	1	1	1
	1	ω	ω^2
	1	ω^2	ω

It is a big theorem that this is a square matrix.

$G = S_3$

	1	(13) (23) (12)	(132) (123)
triv	1	1	1
sign	1	-1	1
2-dim			

$$S_3 \longrightarrow GL_3(\mathbb{C})$$

$$(12) \longmapsto \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad (123) \longmapsto \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

$$\begin{tikzcd} S_3 \arrow[r] \arrow[rr, bend right, "f"'] & GL_3(\mathbb{C}) \arrow[r, "\det"] & \mathbb{C}^\times \end{tikzcd}$$

The image of this map f is $\{-1, 1\}$.

$\ker f = A_3 = \langle (123) \rangle$.

Lemma: $\chi_{V \oplus W} = \chi_V + \chi_W$

$\mathrm{Perm}_{\{1,2,3\}} = \mathrm{triv} \oplus \text{2-dim}$ let us call this standard

~~let us call it "standard"~~

$\chi_{perm} = \chi_{triv} + \chi_{std}$

Now the last line of the character table is:

$2 \quad 0 \quad -1$

Orthogonality Relation

$$\frac{1}{|G|}\sum_{g\in G} \chi_i(g)\chi_j(g^{-1}) = \delta_{ij}$$

where χ_i are ~~irred~~ characters of irred reps.