Lecture 4

Last time we showed Mashke's Thm $\quad (V,\pi)$ rep. of G, $F = \mathbb{C}$

$$V = \underbrace{V_1 \oplus \cdots \oplus V_1}_{a_1} \oplus \underbrace{V_2 \oplus \cdots \oplus V_2}_{a_2} \oplus \cdots$$

↳ V_i irred. subrepn's of V

where $a_i = \dim_{\mathbb{C}} \mathrm{Hom}_G(V_i, V)$

Note: If (V,π) and (W,ϕ) repn's s.t. $\dim_{\mathbb{C}} \mathrm{Hom}_G(U,V) = \dim_{\mathbb{C}} \mathrm{Hom}_G(U,W)$ for all irreducible U of G, then $V \cong W$

Defined the character of V last time:

$$\chi_V : G \longrightarrow \mathbb{C} \quad \text{by } \chi_V = \mathrm{tr}(\pi(g))$$

- $\chi_V(e) = \dim V$
- $\chi_V(g) = \chi_V(hgh^{-1})$, $\quad g, h \in G$

Class function is a function $f: G \to \mathbb{C}$ which is constant of conjugacy classes of G.

First orthogonality relations: $\quad F = \mathbb{C}$

Let V, W be two irred. repn's of G.

Then $\frac{1}{|G|} \sum_{g\in G} \chi_V(g)\overline{\chi_W(g)} = \quad \left(\overline{\chi_W(g)} = \chi_W(g^{-1})\right)$

$$= \begin{cases} 0 & \text{if } V \not\cong W \\ 1 & \text{if } V \cong W \end{cases}$$

Remark: ① If $V \cong W$, then $\chi_V = \chi_W$

② The converse is also true, if V, W are irreducible: if $\chi_V = \chi_W$, then $V \cong W$

Vector space of all class functions (over $\mathbb{C}$)

↳ dim = # conjugacy classes

- $f_1, f_2 \in \mathrm{Class}(G) \equiv$ class functions
- $\langle f_1, f_2 \rangle := \frac{1}{|G|} \sum f_1(g)\overline{f_2(g)}$

Check: $\langle \, , \, \rangle$ is Hermitian

$\langle \, , \, \rangle$ is positive definite

- $\langle f, f \rangle = \frac{1}{|G|} \sum_{g\in G} \|f(g)\|^2 \in \mathbb{R} \quad \geq 0$

- $\langle f, f \rangle = 0 \iff f \equiv 0$

List $V_1, \dots,$ irreducibles of G

$\chi_{V_1}, \dots$ $\leftarrow$ orthonormal set in Class(G)

$\implies$ # irreducibles $\leq$ # conjugacy classes

[Later: let's see when we have equality]

V not necessarily irreducible

$V = \bigoplus_{i=1}^{k} a_i V_i \qquad V_i$ irreducible

$\chi_V = \sum_{i=1}^{k} a_i \chi_{V_i}$ as functions on G

U irreducible $\to \langle \chi_U, \chi_V \rangle = \langle \chi_U, \sum a_i \chi_{V_i} \rangle =$

$= \sum a_i \langle \chi_U, \chi_{V_i} \rangle =$

$= \begin{cases} a_i & \text{if } U \cong V_i \\ 0 & \text{if } U \text{ does not appear in decomposition} \end{cases}$

$\langle \chi_{V_i}, \chi_V \rangle = a_i$

Moreover, $\langle \chi_V, \chi_V \rangle = \langle \sum a_i \chi_{V_i}, \sum a_i \chi_{V_i} \rangle =$

$= \langle a_1 \chi_{V_1}, a_1 \chi_{V_1} \rangle + \langle a_2 \chi_{V_2}, a_2 \chi_{V_2} \rangle + \dots =$

$= a_1^2 + a_2^2 + \dots$

$\therefore \langle \chi_V, \chi_V \rangle = a_1^2 + \dots + a_k^2$

Corollary: $\langle \chi_V, \chi_V \rangle = 1 \iff V$ irreducible,

since $\sum a_i^2 = 1 \implies \exists!_i$ s.t. $a_i = 1$

E.g. Regular representation:

$X = G$. Let G act on $X = G$ by left multiplication. $tr(\pi(g)) = \# fix(g)$

Let V_{reg} be the corresponding linear rep'n $g(e_{\tilde{g}}) := e_{(g\tilde{g})}$

$$\chi_{V_{reg}}(g) = \#\{\tilde{g} \in G \mid g\tilde{g} = \tilde{g}\} =$$

$$= \#\{\tilde{g} \in G \mid g = e\}$$

So $g \neq e \Rightarrow \chi_{V_{reg}}(g) = 0$!

$$\text{So } \chi_{V_{reg}}(g) = \begin{cases} |G| & \text{if } g = e \\ 0 & \text{if } g \neq e \end{cases}$$

$$\langle \chi_{reg}, \chi_{reg} \rangle = \frac{1}{|G|} \chi_{V_{reg}}(e)^2 = \frac{|G|^2}{|G|} = |G|$$

Let U be a irreducible representation of G:

$$\langle \chi_U, \chi_{V_{reg}} \rangle = \frac{1}{|G|} \sum_{g \in G} \chi_U(g) \overline{\chi_{reg}(g)}$$

$$= \frac{1}{|G|} \chi_U(e) \cdot |G| = \chi_U(e) \cancel{|G|}$$

$$= \dim U$$

$$\Rightarrow V_{reg} = \bigoplus_{\substack{\text{all}\\ \text{irreducibles}}} (\dim U) \cdot U$$

Corollary: $\underset{\underset{|G|}{\shortparallel}}{\dim V_{reg}} = \sum_{\text{irreducibles}} (\dim U)(\dim U)$

$$\Longrightarrow |G| = \sum_{\substack{\text{rep.}\\ \text{irred.}}} (\dim U)^2$$

	Cl_1	Cl_2	Cl_3	$\cdots$
χ_1				
χ_2				
$\vdots$				

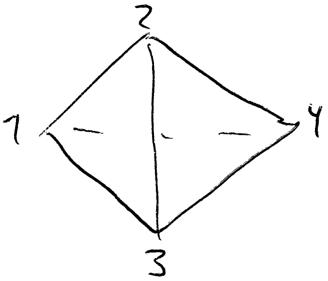
Character table

$\chi_i(g)$ for $g \in Cl_j$

$\#$ rows $\leq \#$ columns

E.g. A_4 = alternation gp. on 4 letters

or View A_4 as the rotational symmetries of



G

$G \hookrightarrow S_4$

$|G| = 3 \cdot 4 = 12 \Rightarrow G \cong A_4$

Each $g \in G = A_4 \rightsquigarrow 3\times 3$ orthogonal matrix of det 1 ($SO(3)$)

- Identity
- Rotation about ~~a~~ vertex by $120°$ $\Rightarrow$ $4 \cdot 2 = 8$
- $(12)(34)$
  $(13)(24)$ $=$ rotations by $180°$ about axis through opposite eges
  $(14)(23)$

identity $\to$ 3

rot. by $180°$ $\to$ eigenvalues are $1+(-1)+(-1) = -1$

rot by $120°$ $\to$ eigenvalues are $1+\omega+\omega^2 = 0$
cube roots

	1	rot by $180°$	rot by $120°$	rot by $-120°$
1	1	1	1	1
	1	1	ω	ω^2
χ_3	1	1	ω^2	ω
$\chi =$	3	-1	0	0

irre

$\langle \chi, \chi \rangle = \frac{1}{12}\{3^2 + 3\cdot(-1)^2 + 8(0^2)\} = \frac{1}{12}\cdot 12 = 1$

- $|G| = 12 = 3^2 + \underbrace{\overbrace{\qquad}^{?}}_{=3\times 1^2}$

- $\langle \chi_3, \chi_3 \rangle = \frac{1}{12}\{4\cdot 1 + \underbrace{4\cdot 1}_{\omega\bar{\omega}} + \underbrace{4\cdot 1}_{\omega^2\cdot\overline{\omega^2}}\}$