$|G| = 12 = 3^2 + 1^2 + 1^2 + 1^2$

1			
1			
1			
3	-1	0	0

$A_4 \to A_4 / \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \cong C_3 \longrightarrow$

	1	1	1	1
	1	1	ω	ω^2
χ_3	1	1	ω^2	ω
3	3	-1	0	0

$\langle \chi_3, \chi_3 \rangle = \frac{1}{12} \{ 4 \cdot 1 + \underset{\omega \cdot \bar{\omega}}{4 \cdot 1} + \underset{\frac{\omega^2}{\omega^2}}{4 \cdot 1} \}$

19/07/13

(V, π) rep of G, $F = \mathbb{C}$

Maschke's thm $\Rightarrow V = \oplus a_i V_i$

$V_i \subset V$ irred subrep.

Schur's lemma $\Rightarrow a_i = \dim_{\mathbb{C}} \mathrm{Hom}_G(V_i, V)$

$V \cong W \Leftrightarrow \dim \mathrm{Hom}_G(U,V) = \dim \mathrm{Hom}_G(U,W) \quad \forall$ irred U

$\chi_V : G \longrightarrow \mathbb{C}$

$$\chi_V(g) = tr(\Pi(g))$$

$\#$ irred reps of $G \leq \#$ conj. classes of G
because of 1^{st} orthog relations
(equality holds)

1^{st} Orthog relations: V, W reps

$$\langle \chi_V, \chi_W \rangle := \frac{1}{|G|} \sum_{g \in G} \chi_V(g) \underbrace{\overline{\chi_W(g)}}_{= \chi_W(g^{-1})}$$

$$V, W \text{ irred} \Rightarrow \langle \chi_V, \chi_W \rangle = \begin{cases} 1 & \text{if } V \cong W \\ 0 & \text{otherwise} \end{cases}$$

If G acts on a set X then we get V_X permutation rep.

$$\# \text{ orbits of } G \text{ on } X = \frac{1}{|G|} \sum_{g \in G} |Fix(g)|$$

$$\dim(V_X^G) = \frac{1}{|G|} \sum_{g \in G} \underbrace{tr(\Pi_X(g))}_{= \chi_{V_X}(g)}$$

The trivial rep has character $\chi_{triv} \equiv 1$

Burside / Frobenius says

$$\dim(V_X^G) = \langle \chi_{V_X}, 1\rangle$$

$\left(\text{Recall: Any rep } V \text{ of } G,\ V^G = \{v \in V \mid g\cdot v = v \ \forall g \in G\} \right)$

"invariants"

$$\text{Maschke's thm} \Rightarrow V = \underbrace{\overbrace{\mathbb{C} \oplus \cdots \oplus \mathbb{C}}^{V^G}}_{a} \oplus \text{ other irred stuff (not trivial)}$$

trivial

V^G = sum of all trivial ~~rep~~ subreps in V.

$\dim V^G$ = multiplicity of trivial rep in V

<u>Prop:</u> If V is any rep of G, then $\dim V^G = \frac{1}{|G|} \sum_{g\in G} \chi_V(g) =$

$= \langle \chi_V, 1 \rangle$

Pf: Let $p: V \longrightarrow V$

$$p(v) = \frac{1}{|G|} \sum_{g \in G} g\cdot v$$

claim: $\operatorname{Im} p = V^G$

$v \in V$, $\tilde{g} \in G$: $\quad \tilde{g}\cdot p(v) = \frac{1}{|G|} \sum_{g\in G} (\tilde{g}\cdot g) v = \frac{1}{|G|} \sum_{g \in G} g\cdot v =$

$= p(v)$

$\Downarrow$

$p(v) \in V^G$

Conversely, if $v \in V^G$

then $p(v) = \frac{1}{|G|}(|G|\cdot v) = v \Rightarrow v \in \operatorname{Im} p$

since $g\cdot v = v \ \forall g \in G$

In fact, p is the identity on V^G

$V = V^G \oplus \ker p$

Compute the trace of p on V:

a matrix for p will look like that

$$\begin{array}{cc} & \begin{array}{cc} V^G & \ker p \end{array} \\ \begin{array}{c} V^G \\ \ker p \end{array} & \left(\begin{array}{c|c} \begin{matrix} 1 & & \\ & \ddots & \\ & & 1 \end{matrix} & 0 \\ \hline 0 & 0 \end{array}\right) \end{array}$$

$tr(\rho) = \dim V^G$

and

$$tr(\rho) = \frac{1}{|G|} \sum_{g \in G} tr(g) = \frac{1}{|G|} \sum_{g \in G} \chi_V(g)$$

<u>Prop:</u> Let V, W be repres.

$$< \chi_V, \chi_W > = \dim_{\mathbb{C}} Hom_G(V, W).$$

Pf: Recall $Hom(V,W)$ can be made into a rep by:

$f \in Hom(V, W)$

$$(g \cdot f)(\underset{V}{\underset{\cap}{v}}) := g \cdot \left(f(g^{-1} \cdot v)\right)$$

$$Hom_G(V, W) = \left(Hom(V, W)\right)^G$$

What is the character of Hom(V,W) as a G-rep?

Fix $g \in G$

Take an eigenbasis of g on V

$$v_1, \dots, v_l \longrightarrow \lambda_1, \dots, \lambda_l$$

Take the same for g on W

$$w_1, \dots, w_m \longrightarrow \mu_1, \dots, \mu_m$$

Construct a basis for Hom(V,W):

$$f_{ij} : v_i \longrightarrow w_j$$

and

$$v_k \longrightarrow 0 \quad \text{when } k \neq i$$

Ex: Compute the trace of g with respect to this basis and you get $\overline{\chi_V(g)} \cdot \chi_W(g)$

<u>Prop:</u> $\chi_{Hom(V,W)} = \overline{\chi_V} \cdot \chi_W$

$$\boxed{\chi_{V^*} = \chi_{Hom(V,\mathbb{C})} = \overline{\chi_V}}$$

trivial

Pf: $\dim_{\mathbb{C}} Hom_G(V,W) = \dim Hom(V,W)^G$

$= \langle \chi_{Hom(V,W)}, 1 \rangle$ by 1^{st} prop.

integer

$= \langle \overline{\chi_V} \cdot \chi_W, 1 \rangle = \langle \chi_W, \chi_V \rangle = \overline{\langle \chi_V, \chi_W \rangle} =$

$= \langle \chi_V, \chi_W \rangle .$

<u>Pf of 1^{st} orthog. relations:</u>

V, W irred then $\langle \chi_V, \chi_W \rangle = \dim_{\mathbb{C}} \mathrm{Hom}_G(V,W) =$

$$= \begin{cases} 1 & \text{if } V \cong W \\ 0 & \text{if } V \not\cong W \end{cases}$$ by Schur's lemma.

<u>Character table for S_5:</u> $\qquad |S_5| = 120 = 5!$

	1	10 elem. $(1,2)$	20 $(1\,2\,3)$	30 $(1\,2\,3\,4)$	24 $(1\,2\,3\,4\,5)$	20 $(1\,2\,3)(4\,5)$	15 $(1\,2)(3\,4)$
triv	1	1	1	1	1	1	1
sign	1	-1	1	-1	1	-1	1
std	4	2	1	0	-1	-1	0
$\chi_{std}\chi_{sign}$	4	-2	1	0	-1	1	0 $\leftarrow (\text{std})^*$
5-dim	5	-1	-1	1	0	-1	-1
5-dim$\cdot\chi_{sign}$	5	1	-1	-1	0	1	-1
	6	0	0	0	1	0	-2

$X = \{1, 2, \dots, 5\}$

$$V_X = \underset{\langle e_1 + \dots + e_5 \rangle}{\underset{\shortparallel}{\text{triv}}} \oplus \underset{\{\sum c_i e_i \,/\, \sum c_i = 0\}}{\underset{\shortparallel}{\text{standard}}}$$

$$\begin{array}{rccccccc}
 & & (1,2) & & & & & \\
\chi_{V_X} = & 5 & 3 & 2 & 1 & 0 & 0 & 1 \\
-\ \text{triv} = & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
\hline
 & 4 & 2 & 1 & 0 & -1 & -1 & 0
\end{array}$$

$$\langle \chi_{std}, \chi_{std} \rangle = \frac{1}{120}\left(4^2 + 10 \cdot 2^2 + 20 \cdot 1^2 + 24 \cdot (-1)^2 + 20(-1)^2\right)$$

$$= \frac{1}{120}\left(16 + 40 + 20 + 24 + 20\right) = 1$$

Y = 2-element subsets of $\{1, \dots, 5\}$

S_5 acts transitively

$$Y \cong S_5 / \underset{S_2 \times S_3}{\underset{\shortparallel}{\text{stabilizer}}}$$

$|Y| = 10$

V_Y is ~~10-dimens~~ 10-dim.

$$\chi_{V_Y} = 10 \quad 4 \quad 1 \quad 0 \quad 0 \quad 1 \quad 2$$

{1,2}
{3,4}
{3,5}
{4,5}

$$< \chi_{V_Y}, \chi_{V_Y} > = \frac{1}{120} \{ 10^2 + 10.4^2 + 20 \cdot 1^2 + 20 \cdot 1^2 + 15 \cdot 2^2 \} =$$

$$= 3$$

$V = \oplus a_i V_i \Rightarrow < \chi_V, \chi_V > = \sum a_i^2$

V_Y = sum of 3 distinct irred reps.

$= \text{triv} \oplus$ ______ $\oplus$ ______

$$\chi_{V_Y} - \text{triv} = 9 \quad 3 \quad 0 \quad -1 \quad -1 \quad 0 \quad 1$$

$< \chi_{std}, \chi_{V_Y} - \text{triv} > = 1$ check

$\underset{10\text{-dim}}{V_Y} = \underset{1\text{-dim}}{\text{triv}} \oplus \underset{4\text{-dim}}{\text{std}} \oplus$ 5-dim

and $< \chi_{std}, \chi_{5\text{-dim}} > = 1$

$F = \mathbb{C}$

$\dim W \mid |G|$ if W irred $\longrightarrow \frac{|G|}{\dim W} \in \mathbb{Q}$

(algebraic integers) it is an alg integer

$\Downarrow$

$\frac{|G|}{\dim W} \in \mathbb{Z}$

S_n : all characters are integers.

formulas for the character values