Eric Sommers - 2^{nd} lecture

G finite group

(V,π), $\pi : G \longrightarrow GL(V) \cong GL_n(F)$

def. of representation

$W \subset V$ is a <u>subrepresentation</u> means W is a subspace and $g.w \in W$ $\forall g \in G$, $\forall w \in W$

<u>def</u>: V is <u>irreducible</u> (simple) if $\{0\}$ and V are the only subreps of V $(V \neq 0)$

<u>completely reducible</u>: V is completely reducible means $V = W_1 \oplus \dots \oplus W_k$ where $W_i \subset V$ irreducible, subrep. (v. space direct sum)

<u>Maschke's thm</u>: If char $F = 0$ or char $F \nmid |G|$, then V is completely reducible.

$F = \mathbb{C}$

proof: $(\, , \,) : V \times V \longrightarrow \mathbb{C}$ Hermitian form

$(v_1 + v_2, w) = (v_1, w) + (v_2, w)$

$(\lambda v, w) = \lambda (v, w)$, $\lambda \in \mathbb{C}$

$(v, \lambda w) = \bar{\lambda} (v,w)$

$(v,w) = \overline{(w,v)}$

positive definite: $(v,v) \geq 0$ and $(v,v) = 0$ iff $v = 0$.

$(e_i, e_j) = \delta_{ij}$, extend sesquilinear

$(x_1, \dots, x_n) \cdot (y_1, \cdots, y_n) = \sum x_i \bar{y}_i$

[fact: all such forms are equivalent up to change of basis in V]

<u>def</u>: $T : V \longrightarrow V$; if $(T.v, T.w) = (v,w)$ $\forall v, w$, we say T is unitary

A unitary matrix if it preserves standard Hermitian form $\Leftrightarrow A\bar{A}^t = I$

$U_n := \{$all such $A\}$ unitary gp

$\cup$

$O_n := \{$all $A \in GL_n(\mathbb{R}) : AA^t = I\}$ compact Lie group

$\langle v, w \rangle := \sum_{g \in G} (g.v, g.w)$ start w/ std p.d. Hermitian form

exer: $\langle \, , \rangle$ is a pos. def. Hermitian form on V

fact: $\langle \, , \rangle$ is G-invariant

ie $\langle \pi(g).v, \pi(g).w \rangle = \langle v, w \rangle$ $\forall g \in F$, $\forall v, w \in V$

ie $\pi(g) \in U_n$ (up to change of basis)

$\tilde{g} \in G$ $v, w \in W$

$\langle \tilde{g}.v, \tilde{g}w \rangle = \sum_{g \in G} (g\tilde{g}.v, g\tilde{g}.w)$

$= \sum_{g \in G} (g.v, g.w) = \langle v, w \rangle$

Let $W \subset V$ subrep.

$W^\perp := \{v \in V \mid \langle v, w \rangle = 0 \quad \forall w \in W\}$

$V = W \oplus W^\perp$ as v. spaces (pos. def. Herm.)

$W^\perp$ is a subrep of V

$g \in G$, $v \in W^\perp$ WANT: $g.v \in W^\perp$

$\langle g.v, w \rangle = \langle v, g^{-1}.w \rangle = 0$

$w \in W$ $\quad v \in W^\perp$, $g^{-1}.w \in W$

$v \in W^\perp$

$\Rightarrow g.v \in W^\perp$ and $W^\perp$ subrep.

finish: • V irred. ✓

• V reducible $\Rightarrow 0 \subsetneq W \subsetneq V$ subrep.

$\Rightarrow V = W \oplus W^\perp$

nonzero + proper

• iterate ($\dim_{\mathbb{C}} V < \infty$)

$V \xrightarrow[p]{} W$, $p: V \to V$, $W \subset V$

$\mathrm{im}\, p = W$
$p^2 = p$ $\qquad \tilde{p} = \frac{1}{|G|} \sum_{g \in G} g \circ p \circ g^{-1}$

$\mathrm{im}\, \tilde{p} = W$, $\ker \tilde{p} \oplus W = V$

fact: $\pi(g)$ is diagonalizable $\forall g \in G$

pf 1: $\pi(g) \in U_n \Rightarrow \pi(g)$ diag.
(normal operator)

pf 2: $H = \langle g \rangle$ cyclic gp. $\subset G$
(abelian)

$\Rightarrow$ the irred. repns of H are 1-dim'l

$g \curvearrowright V$ repn of H (restriction)

$V_{|H} = \langle v_1 \rangle \oplus \langle v_2 \rangle \oplus \cdots \oplus \langle v_n \rangle$
subreps

$g.v_i = \lambda_i v_i$, ie v_i is an eigenvector for $\pi(g)$.

recall: $\pi(g)$ diag. means V has a basis of eigenvecs.

$\pi(g) = \begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{pmatrix}$ w.r.t. this basis
$A\bar{A}^t = I$

$|\lambda_i| = 1$ since unitary
in fact, $\pi(g)^m = 1 \Rightarrow \lambda_i^m = 1$

$\mathrm{tr}(\pi(g)) = \sum \lambda_i$ = sum of e.values
algebraic integer

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V, W reps $\to V \oplus W$ rep
$g(v, w) = (g.v, g.w)$

F v.space
$\hookrightarrow \mathrm{Hom}(V, W) = \{\theta : V \to W$ linear$\}$

Make $\mathrm{Hom}(V, W)$ into a rep of G

$\theta \in \mathrm{Hom}(V, W)$, $g.\theta = ?$

$g.\theta(v) \overset{\mathrm{def}}{=} g\theta(g^{-1}v)$ $\qquad \boxed{(gh).\theta = g.(h\theta)}$

$[(gh).\theta](v) = (gh).\theta((gh)^{-1}.v)$ $\qquad (gh)^{-1} = h^{-1}g^{-1}$

$(g(h.\theta))(v) = g.[(h\theta)(g^{-1}v)]$
$= g.[h.\theta(h^{-1}(g^{-1}v))] \ldots$

check details

$\mathrm{Hom}(V, W)$ rep of G
$\cup$
$\mathrm{Hom}_G(V, W) = \left\{ \theta : V \to W \text{ linear}, \ \phi(g) \circ \theta = \theta \circ \pi(g) \text{ for all } g \in G \right\}$

(V, π), (W, ϕ)

maps between rep (V, π) (W, ϕ)
means $T \in \mathrm{Hom}_G(V, W)$

homomorphisms of reps.

T is an isomorphism or equivalence if $T \in \mathrm{Hom}_G(V, W)$ and it's invertible (ie. isom. of v. spaces)

U is a rep of G
$U^G := \{u \in U \mid g.u = u \ \forall g \in G\}$

exerc: $\mathrm{Hom}(V, W)^G = \mathrm{Hom}_G(V, W)$

3^{rd} lecture

Schurs lemma

1. V, W irreducible and $V \not\cong W$ then $\mathrm{Hom}_G(V, W) = \{0\}$ (any F)

2. V irred., $\mathrm{End}_G(V) = \mathrm{Hom}_G(V, V)$ is a ring under composition $T \circ T'$ called an F-algebra. Then $\mathrm{End}_G(V)$ is a division ring.

[fields $\subset$ div. rings; an example of a div. ring which isn't a field is $\mathbb{H}$ = quaternions; $x \in \mathbb{H} - \{0\} \Rightarrow x^{-1}$ exists; in general $xy \neq yx$]

3. V irred., $F = \mathbb{C}$, $\mathrm{Hom}_G(V, V) \cong \mathbb{C}$ consists of $\{\lambda \cdot \mathrm{id}_V \mid \lambda \in \mathbb{C}\}$

Pf: ① $\phi: V \to W$

$\ker\phi \subset V$ subrep

$\mathrm{im}\,\phi \subset W$ subrep

$v \in \ker\phi$ and $g \in G$ WANT: $g.v \in \ker\phi$

$\phi(g.v) = g.\phi(v) = g.0 = 0 \iff g.v \in \ker\phi$

$\Rightarrow \ker\phi \subset V$ subrep.

$\ker\phi = 0$ or V

$\mathrm{im}\,\phi = 0$ or W

If $\ker\phi = V$ or $\mathrm{im}\,\phi = 0$, then $\phi \equiv 0$

If $\phi \not\equiv 0$, then $\ker\phi = 0$ and $\mathrm{im}\,\phi = W$

$\iff$ ϕ isom. of v. spaces

$\iff$ " " reps

$\Rightarrow V \cong W$ $\Rightarrow \Leftarrow$

② from ① if $\phi \neq 0$, then $\exists$ an isomorphism $\Rightarrow$ ϕ is invertible in $\mathrm{End}\,G(V)$ $\Rightarrow$ $\mathrm{End}_G(V)$ is a division ring.

③ $F = \mathbb{C}$ and $\phi: V \to V$ $\exists v \neq 0$ st $\phi(v) = \lambda v$ since $\lambda \in \mathbb{C}$ "eigenvector"

$\iff \ker(\phi - \lambda_0 \mathrm{id}_V) \neq \{0\}$ $\quad$ V irred.

$\ker(\phi - \lambda_0 \mathrm{id}_V) \subset V$ subrep

$\Rightarrow \ker(\phi - \lambda \mathrm{id}_V) = V$

($\mathrm{End}_G(V)$ is closed under $+$)

given $\phi \in \mathrm{End}_G(V)$,

$\lambda\, \mathrm{id}_V \in \mathrm{End}_G(V) \supset F \cong \{\lambda \cdot \mathrm{id}_V\}$

$g.(\lambda\, \mathrm{id}_V(v)) = g(\lambda . v) = \lambda(g.v)$
$= (\lambda\, \mathrm{id}_V)(g.v)$ $\quad$ g linear

$\to \phi - \lambda\, \mathrm{id}_V \equiv 0$

$\boxed{\phi \equiv \lambda\, \mathrm{id}_V}$

$\mathbb{Q}(\omega)$ $\quad \omega = \sqrt[3]{-1}$ $\quad \circlearrowleft G$
|
$\mathbb{Q}$ $\quad$ rep^n of C_6 on $\mathbb{Q}(\omega)$ given by left mult. of $-\omega$

[$F = \mathbb{Q}$ rep^n]

thus 2-dim rep. is irred. over $\mathbb{Q}$. why?

schur $\Rightarrow$ $\mathrm{End}_G(\underbrace{\mathbb{Q}(\omega)}_{=V})$ $\qquad \begin{pmatrix} 0 & 1 \\ -1 & 1 \end{pmatrix}$

$\mathbb{Q}$
$\cap$

$\mathrm{End}_G(V \cong \mathbb{Q}^2) \subset \mathrm{End}(\mathbb{Q}^2) = M_2(\mathbb{Q})$ $\qquad \begin{vmatrix} -\lambda & 1 \\ -1 & 1-\lambda \end{vmatrix} = \lambda^2 - \lambda + 1$

$\mathbb{Q} = \left\{ \begin{pmatrix} \lambda & \\ & \lambda \end{pmatrix} \right\} \subset \underset{\mathbb{Q}(\omega)}{?} \subsetneq M_2(\mathbb{Q})$

$\mathrm{End}_G(V) \cong \mathbb{Q}(\omega) \leftarrow$ field

$\Rightarrow \left\{ \begin{pmatrix} 1 & \\ & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ -1 & 1 \end{pmatrix} \right\}$

Maschke's

$$V = \underbrace{(V_1 \oplus \dots \oplus V_1)}_{a_1} \oplus \underbrace{(V_2 \oplus \dots \oplus V_2)}_{a_2} \oplus \dots \oplus \underbrace{(V_k \oplus \dots \oplus V_k)}_{a_k}$$

V_i are irred. subreps, mutually non isomorphic.

W an arbitrary irred. rep.

$\mathrm{Hom}_G(W, V) = \mathrm{Hom}_G(W, \bigoplus_i a_i V_i)$

as v. spaces $\to$ $= \bigoplus_i a_i \mathrm{Hom}_G(W, V_i)$ "check"

$= \begin{cases} \oplus\, a_j \mathrm{Hom}_G(W, V_j) & \text{where } W \cong V_j \\ 0 & \text{if no such } j \end{cases}$

$\Rightarrow \dim_{\mathbb{C}} \mathrm{Hom}_G(W, V) = a_j$

✓ if $W \cong V_j$ appears in decomp.

counting multiplicity of W in rep. V

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$$V = \underbrace{(V_1 \oplus \dots \oplus V_1)}_{W_1} \oplus \dots \oplus \underbrace{(V_j \oplus \dots \oplus V_j)}_{W_j} \oplus \dots$$

claim: $W_i = \sum U$ where $U \subset V$ subrep and $U \cong V_i$

ie W_i is uniquely determined

" V_i - isotypic components of V ".

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character

$\chi: G \to \mathbb{C}^\times$ is called character

$\hat{G} := \mathrm{Hom}(G, \mathbb{C}^\times)$ gp. homom.

↑ abel. gp.

If (V, π) is an arbitrary rep, then

def: the (linear) character of V, denoted $\chi_V : G \to \mathbb{C}$, is by def
$$\chi_V(g) = tr(\pi(g)) = \sum_{i=1}^{n} \lambda_i$$

facts:

1. $V \cong W \Rightarrow \chi_V = \chi_W$

2. $g = xhx^{-1}$ in G, $g, h, x \in G$

ie g, h in same conj. class in G

[$\pi(g) = \pi(xhx^{-1})$
$= \pi(x)\pi(h)\pi(x)^{-1}$

$\pi(g)$ and $\pi(h)$ are conjugates]

$\Rightarrow tr(\pi(g)) = tr(\pi(h))$

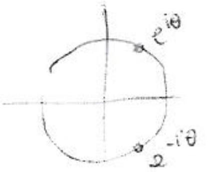
$\Rightarrow \chi_V(g) = \chi_V(h)$

ie χ_V constant on conjugacy classes
"class functions"

3. V is 1-diml: χ_V agrees w/ existing defn.

4. $\chi_V(g) = \sum \lambda_i$ where $\pi(g) \sim \begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{pmatrix}$

$$\begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{pmatrix}^{-1} = \begin{pmatrix} \lambda_1^{-1} & & \\ & \ddots & \\ & & \lambda_n^{-1} \end{pmatrix} = \begin{pmatrix} \bar\lambda_1 & & \\ & \bar\lambda_2 \ddots & \\ & & \bar\lambda_n \end{pmatrix}$$



$tr(\pi(g^{-1})) = \sum \bar\lambda_i$
$= \overline{\sum \lambda_i} = \overline{tr(\pi(g))}$

$\chi_V(g^{-1}) = \overline{\chi_V(g)}$

character table for G

$G = C_3$

	1	x	x^2 — conjugacy classes
triv.	1	1	1
	1	ω	ω^2
	1	ω^2	ω

irred reps

rows and columns orthogonal

$1 + \omega + \omega^2 = 0$

big thm: square matrix

discrete Fourier transform

$G = S_3$

	1	(12) (13) (23)	(123) (132)
triv	1	1	1
sign	1	-1	1
2-d	2	0	-1

$S_3 \to \{1,2,3\}$ $\quad (12) \to \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ det $= -1$

$(123) \to \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$ det $=$

$\phi : S_3 \hookrightarrow GL_3(\mathbb{C}) \to \mathbb{C}^\times$

$f(AB) = f(A)f(B)$, $f = \det$

$\{\pm 1\}$

$\ker\phi = A_3 = \langle (123) \rangle$

$V \oplus W \quad \chi_{V \oplus W} = \chi_V + \chi_W \quad \left(\begin{array}{c|c} A & 0 \\ \hline 0 & B \end{array}\right)$

$tr = tr(A) + tr(B)$

Perm $\{1,2,3\}$ = trivial $\oplus$ 2-diml
= standard

$\chi_{perm} = \chi_{triv} + \chi_{std}$

$$\frac{1}{|G|} \sum_{g \in G} \chi_i(g)\, \chi_j(g^{-1}) = \delta_{ij}$$

where χ_i are the characters of irred. reps