Problem Set 1 for Etingof's course

It would be very helpful that you read sections 1.1 through 1.8 of the lecture notes available at:

`http://ocw.mit.edu/courses/mathematics/18-712-introduction-to-representation-theory-fall-2010/lecture-notes/`

(you'll find a link in the "Exercises and Notes" section on the webpage of the conference). The problems below are problems 1.33 and 1.38 in the above text.

1. The *path algebra* P_Q of a quiver is the algebra whose basis is formed by oriented paths in Q, including the trivial paths p_i corresponding to the vertices of Q, and multiplication is by concatenation of paths: ab is the path obtained by first tracing b and then a. If two paths cannot be concatenated, the product is defined to be zero.

Show that the algebra P_Q is generated by p_i, for $i \in I$ and a_h for $h \in E$ (where I is the set of vertices and E is the set of edges) with the defining relations:

1. $p_i^2 = p_i, p_i p_j = 0$ for $i \neq j$,
2. $a_h p_{h'} = a_h, a_h p_j = 0$ for $j \neq h'$
3. $p_{h''} a_h = a_h, p_i a_h = 0$ for $i \neq h''$

2. Let A be a $\mathbb{Z}_+$-graded algebra, i.e. $A = \oplus_{n \geq 0} A[n]$, and $A[n] \cdot A[m] \subset A[n+m]$. If $A[n]$ is finite dimensional, it is useful to consider the Hlbert series $h_A(t) = \sum \dim A[n] t^n$ (the generating function of dimensions of $A[n]$). Often this series converges to a rational function, and the answer is written in the form of such a function. For example, if $A = k[x]$ and $\deg(x^n) = n$, then

$$h_A(t) = 1 + t + t^2 + \ldots t^n + \ldots = \frac{1}{1-t}$$

Find the Hilbert series of:

(a) $A = k[x_1, \ldots, x_m]$ (where the grading is by degree of polynomials);

(b) $A = k\langle x_1, \ldots, x_m \rangle$ (the grading is by length of words);

(c) A is the exterior (=Grassmann) algebra $\Lambda_k[x_1, \ldots, x_m]$, generated over some field k by $x_1, \ldots, x_m$ with the defining relations $x_i x_j + x_j x_i = 0$ and $x_i^2 = 0$ for all i, j (the grading is by degree).

(d) A is the path algebra P_Q of a quiver Q (the grading is defined by $\deg(p_i) = 0$, $\deg(a_h) = 1$). *Hint: The closed answer is written in terms of the adjacency matrix M_Q of Q.*