5 Quiver Representations

5.1 Problems

Problem 5.1. Field embeddings. *Recall that $k(y_1, ..., y_m)$ denotes the field of rational functions of $y_1, ..., y_m$ over a field k. Let $f : k[x_1, ..., x_n] \to k(y_1, ..., y_m)$ be an injective k-algebra homomorphism. Show that $m \geq n$. (Look at the growth of dimensions of the spaces W_N of polynomials of degree N in x_i and their images under f as $N \to \infty$). Deduce that if $f : k(x_1, ..., x_n) \to k(y_1, ..., y_m)$ is a field embedding, then $m \geq n$.*

Problem 5.2. Some algebraic geometry.

Let k be an algebraically closed field, and $G = GL_n(k)$. Let V be a polynomial representation of G. Show that if G has finitely many orbits on V then $\dim(V) \leq n^2$. Namely:

(a) Let $x_1, ..., x_N$ be linear coordinates on V. Let us say that a subset X of V is Zariski dense if any polynomial $f(x_1, ..., x_N)$ which vanishes on X is zero (coefficientwise). Show that if G has finitely many orbits on V then G has at least one Zariski dense orbit on V.

(b) Use (a) to construct a field embedding $k(x_1, ..., x_N) \to k(g_{pq})$, then use Problem 5.1.

(c) generalize the result of this problem to the case when $G = GL_{n_1}(k) \times ... \times GL_{n_m}(k)$.

Problem 5.3. Dynkin diagrams.

Let Γ be a graph, i.e., a finite set of points (vertices) connected with a certain number of edges (we allow multiple edges). We assume that Γ is connected (any vertex can be connected to any other by a path of edges) and has no self-loops (edges from a vertex to itself). Suppose the vertices of Γ are labeled by integers $1, ..., N$. Then one can assign to Γ an $N \times N$ matrix $R_\Gamma = (r_{ij})$, where r_{ij} is the number of edges connecting vertices i and j. This matrix is obviously symmetric, and is called the adjacency matrix. Define the matrix $A_\Gamma = 2I - R_\Gamma$, where I is the identity matrix.

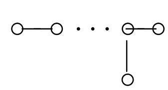
Main definition: Γ *is said to be a Dynkin diagram if the quadratic from on $\mathbb{R}^N$ with matrix A_Γ is positive definite.*

Dynkin diagrams appear in many areas of mathematics (singularity theory, Lie algebras, representation theory, algebraic geometry, mathematical physics, etc.) In this problem you will get a complete classification of Dynkin diagrams. Namely, you will prove

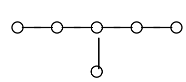
Theorem. Γ *is a Dynkin diagram if and only if it is one on the following graphs:*

- A_n :

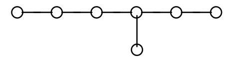
- D_n:



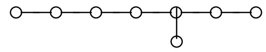
- E_6 :



- E_7 :



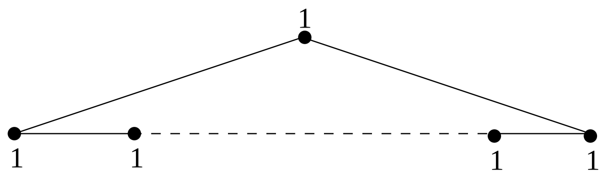
- E_8 :



(a) Compute the determinant of A_Γ where $\Gamma = A_N, D_N$. (Use the row decomposition rule, and write down a recursive equation for it). Deduce by Sylvester criterion[7] *that A_N, D_N are Dynkin diagrams.*[8]

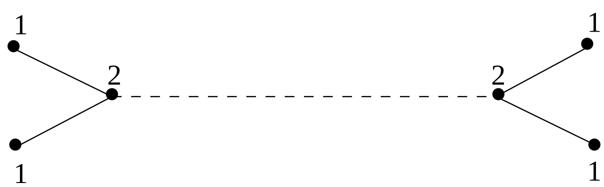
(b) Compute the determinants of A_Γ for E_6, E_7, E_8 (use row decomposition and reduce to (a)). Show they are Dynkin diagrams.

(c) Show that if Γ is a Dynkin diagram, it cannot have cycles. For this, show that $det(A_\Gamma) = 0$ for a graph Γ below [9]

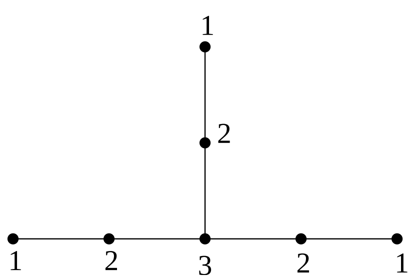


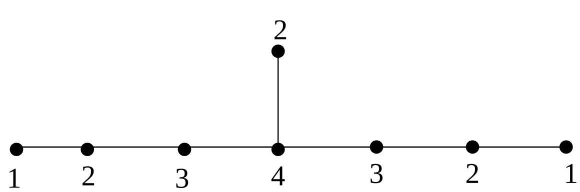
(show that the sum of rows is 0). Thus Γ has to be a tree.

(d) Show that if Γ is a Dynkin diagram, it cannot have vertices with 4 or more incoming edges, and that Γ can have no more than one vertex with 3 incoming edges. For this, show that $det(A_\Gamma) = 0$ for a graph Γ below:



(e) Show that $det(A_\Gamma) = 0$ for all graphs Γ below:

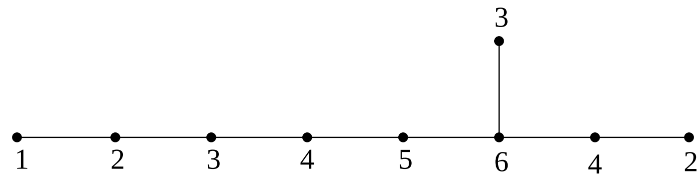




[7]Recall the Sylvester criterion: a symmetric real matrix is positive definite if and only if all its upper left corner principal minors are positive.

[8]The Sylvester criterion says that a symmetric bilinear form $(,)$ on $\mathbb{R}^N$ is positive definite if and only if for any $k \le N$, $\det_{1\le i,j\le k}(e_i, e_j) > 0$.

[9]Please ignore the numerical labels; they will be relevant for Problem 5.5 below.



(f) Deduce from (a)-(e) the classification theorem for Dynkin diagrams.

(g) A (simply laced) affine Dynkin diagram is a connected graph without self-loops such that the quadratic form defined by A_Γ is positive semidefinite. Classify affine Dynkin diagrams. (Show that they are exactly the forbidden diagrams from (c)-(e)).

Problem 5.4. *Let Q be a quiver with set of vertices D. We say that Q is of finite type if it has finitely many indecomposable representations. Let b_{ij} be the number of edges from i to j in Q ($i, j \in D$).*

There is the following remarkable theorem, proved by P. Gabriel in early seventies.

Theorem. *A connected quiver Q is of finite type if and only if the corresponding unoriented graph (i.e., with directions of arrows forgotten) is a Dynkin diagram.*

In this problem you will prove the "only if" direction of this theorem (i.e., why other quivers are NOT of finite type).

(a) Show that if Q is of finite type then for any rational numbers $x_i \geq 0$ which are not simultaneously zero, one has $q(x_1, ..., x_N) > 0$, where

$$q(x_1, ..., x_N) := \sum_{i \in D} x_i^2 - \frac{1}{2} \sum_{i,j \in D} b_{ij} x_i x_j.$$

Hint. It suffices to check the result for integers: $x_i = n_i$. First assume that $n_i \geq 0$, and consider the space W of representations V of Q such that $\dim V_i = n_i$. Show that the group $\prod_i GL_{n_i}(k)$ acts with finitely many orbits on $W \oplus k$, and use Problem 5.2 to derive the inequality. Then deduce the result in the case when n_i are arbitrary integers.

(b) Deduce that q is a positive definite quadratic form.

Hint. Use the fact that $\mathbb{Q}$ is dense in $\mathbb{R}$.

(c) Show that a quiver of finite type can have no self-loops. Then, using Problem 5.3, deduce the theorem.

Problem 5.5. *Let $G \neq 1$ be a finite subgroup of $SU(2)$, and V be the 2-dimensional representation of G coming from its embedding into $SU(2)$. Let V_i, $i \in I$, be all the irreducible representations of G. Let r_{ij} be the multiplicity of V_i in $V \otimes V_j$.*

(a) Show that $r_{ij} = r_{ji}$.

(b) The McKay graph of G, $M(G)$, is the graph whose vertices are labeled by $i \in I$, and i is connected to j by r_{ij} edges. Show that $M(G)$ is connected. (Use Problem 3.26)

(c) Show that $M(G)$ is an affine Dynkin graph (one of the "forbidden" graphs in Problem 5.3). For this, show that the matrix $a_{ij} = 2\delta_{ij} - r_{ij}$ is positive semidefinite but not definite, and use Problem 5.3.

Hint. Let $f = \sum x_i \chi_{V_i}$, where χ_{V_i} be the characters of V_i. Show directly that $((2 - \chi_V)f, f) \geq 0$. When is it equal to 0? Next, show that $M(G)$ has no self-loops, by using that if G is not cyclic then G contains the central element $-Id \in SU(2)$.

(d) Which groups from Problem 3.24 correspond to which diagrams?

(e) Using the McKay graph, find the dimensions of irreducible representations of all finite $G \subset SU(2)$ (namely, show that they are the numbers labeling the vertices of the affine Dynkin diagrams on our pictures). Compare with the results on subgroups of $SO(3)$ we obtained in Problem 3.24.

Problem 5.4' Let Q be a connected quiver, and assume that for any dimension vector d, the number of isomorphism classes of representations of Q over a finite field $\mathbb{F}_p$ is independent of p for large enough primes p. Show that Q is a Dynkin diagram of type A, D, or E. *Hint: This requires Problem 5.3.*