Problem Set 3 for "Representations of quivers

The problems below are Exercises 5.39 (a)-(c), 5.40, 5.41 from the Lecture Notes.

1. Let Q_n be the cyclic quiver of length n, i.e. n vertices connected by n oriented edges forming a cycle. Obviously, the classification of indecomposable representations of Q_1 is given by the Jordan normal form theorem. Obtain a similar classification of indecomposable representations of Q_2. In other words, classify pairs of linear operators $A\colon V \to W$ and $B\colon W \to V$ up to isomorphism. Namely:

(a) Consider the following pairs (for $n \geq 1$):

1) $E_{n,\lambda}\colon V = W = \mathbb{C}^n$, A is the Jordan block of size n with eigenvalue $\lambda, B = 1 (\lambda \in \mathbb{C})$.
2) $E_{n,\infty}\colon$ is obtained from $E_{n,0}$ by exchanging V with W and A with B.
3) $H_n\colon V = \mathbb{C}^n$ with basis v_i, $W = \mathbb{C}^{n-1}$ with basis w_i, $Av_i = w_i, Bw_i = v_{i+1}$ for $i < n$, and $Av_n = 0$.
4) K_n is obtained from H_n by exchanging V with W and A with B.

Show that these are indecomposable and pairwise non-isomorphic.

(b) Show that if E is a representation of Q_2 such that AB is not nilpotent, then $E = E' \oplus E''$ where $E'' = E_{n,\lambda}$ for some $\lambda \neq 0$.

(c) Consider the case when AB is nilpotent, and consider the operator X on $V \oplus W$ given by $X(v, w) = (Bw, Av)$. Show that X is nilpotent, and there is a basis consisting of chains (i.e. sequences $u, Xu, X^2u, \ldots, X^{l-1}u$ with $X^l u = 0$) which are compatible with the direct sum decomposition (i.e. for every chain $u \in V$ or $u \in W$). Deduce that (1)-(4) are the only indecomposable representations of Q_2.

2. Let $L \subset \frac{1}{2}\mathbb{Z}^8$ be the lattice of vectors where the coordinates are either all integers or all half-integers (but not integers), and the sum of all coordinates is an even integer.

(a) Let $\alpha_i = e_i - e_{i+1}, i = 1, \ldots, 6, \alpha_7 = e_6 + e_7, \alpha_8 = -1/2 \sum_{i=1}^{8} e_i$. Shiw that the α_i are a basis of L (over $\mathbb{Z}$).

(b) Show that roots in L (under the usual inner product) form a root system of type E_8 (compute the inner products of the α_i).

(c) Show that the E_7 and E_6 lattices can be obtained as the sets of vectors in the E_8 lattice L where the first two, respectively three, coordinates (in the basis e_i) are equal.

(d) Show that E_6, E_7, E_8 have $72, 126, 240$ roots, respectively (enumerate types of roots in terms of the presentations in the basis e_i, and count the roots of each type).

3. Let V_α be the indecomposable representation of Dynkin quiver Q which corresponds to a positive root α. For instance, if α_i is a simple root, then V_α has a 1-dimensional space at i and 0 everywhere else.

(a) Show that if i is a source then $\mathrm{Ext}^1(V, V_{\alpha_i}) = 0$ for any representation V of Q, and if i is a sink, then $\mathrm{Ext}^1(V_{\alpha_i}, V) = 0$.

(b) Given an orientation of the quiver, find a Jordan-Hölder series of V_α for that orientation.