Representations of quivers I

1) A matrix $m \times n$

$A \longmapsto SAT$; S, T invertible $\longleftarrow$ $\boxed{A: V \to W}$

S changes basis in V
T changes basis in W

what is normal form?

Gaussian elimination

Answer: $$p\left\{\underbrace{\left(\begin{array}{c|c} \overbrace{\begin{matrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{matrix}}^{p} & 0 \\ \hline 0 & 0 \end{array}\right)}_{n}\right\} m$$

$A \longmapsto SAS^{-1} \longleftarrow \boxed{A: V \to V}$

2) A square matrix

Answer: Jordan normal form thm:

$\lambda \in \mathbb{C}$

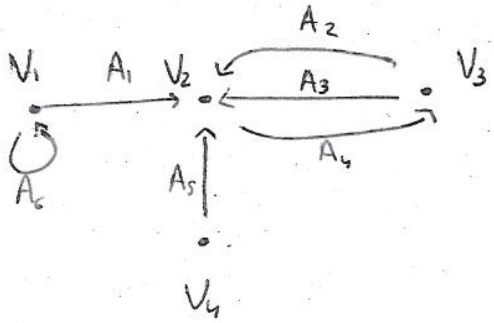
$$J_{\lambda, n} = \begin{pmatrix} \lambda & 1 & & 0 \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ 0 & & & \lambda \end{pmatrix}$$

Every A can be reduced to a direct sum of such "Jordan blocks" $J_{\lambda, n}$

$$\left(\begin{array}{c|c|c|c} J_{\lambda_1 n_1} & 0 & 0 & 0 \\ \hline 0 & J_{\lambda_2 n_2} & 0 & 0 \\ \hline 0 & 0 & \ddots & 0 \\ \hline 0 & 0 & 0 & \ddots \end{array}\right)$$

Def.

A quiver is an oriented graph



Def.

A repr. of a quiver Q is an assignment:

For each vertex i of Q ⇝ vector space V_i (fin. dim.)

For each edge $i \xrightarrow[a]{} j$ ⇝ linear operator $V_i \xrightarrow{\Pi(a)} V_j$

$\boxed{\text{Over some field } F \; . \; V_i \cong F^{d_i} \; ; \; d_i = \dim V_i}$

* Want to classify repr. of quivers up to isomorphis.

Isomorphism:

$(V, \Pi) \xrightarrow{\varphi} (V', \Pi')$

is a collection of ~~isomorf~~ isomorphisms:

$\varphi_i : V_i \xrightarrow{\sim} V_i'$ for all vertices i of Q such that:

$$\begin{tikzcd} V_i \arrow[r, "\Pi(a)"] \arrow[d, "\varphi_i"'] \arrow[dr, phantom, "\circlearrowleft"] & V_j \arrow[d, "\varphi_j"] \\ V_i' \arrow[r, "\Pi'(a)"'] & V_j' \end{tikzcd}$$

⇒ If $V_i \cong F^{d_i}$, $\Pi(a)$ is a $d_j \times d_i$ matrix over F.

Ex. 1 $\bullet \longrightarrow \bullet$ A_2

$$\begin{tikzcd} \underset{F^n}{\bullet} \arrow[loop left, "T"] \arrow[r, "A"] & \underset{F^m}{\bullet} \arrow[loop right, "S"] \end{tikzcd}$$

$m \times n$ matrix

$A \cong A'$ iff $A' = SAT$; S, T invertible.

Gauss. elimination $\Rightarrow$ repns. are parametrized by three integers p, m, n, $p \leq m$, $p \leq n$ and there are finitely many for each dimension

Ex. 2

$$\begin{tikzcd} \bullet \arrow[loop right] \end{tikzcd}$$

Jordan quiver

$$\begin{tikzcd} \overset{F^n}{\bullet} \arrow[loop right, "A"] \end{tikzcd}$$

$A \cong A'$ iff $A' = SAS^{-1}$; S invertible

JNF theorem $\Rightarrow$ all repr. are direct sums of $J_{\lambda,n}$

infinitely many repr. in each dimension.

Consider only connected quivers.

Def.

A quiver Q is of finite type if for any dimensions of spaces at the vertices it has finitely many representations (over $\mathbb{C}$)

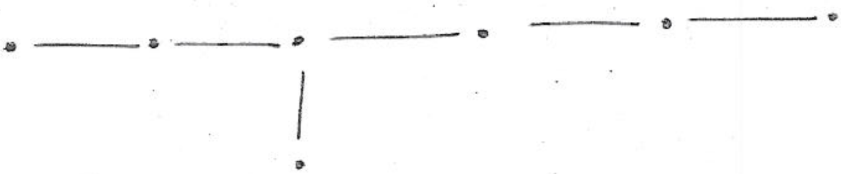
Thm. (Gabriel)

Q is of finite type if and only if it is a simply laced Dynkin diagram:

A_n $n \geq 1$ (number of vertices)

D_n $n \geq 4$

E_6



E_7

E_8

* the property of ~~being~~ finite type is independent on orientation: NONTRIVIAL!!

Direct sum:

$$(V,\Pi)\oplus(V',\Pi') = (V\oplus V', \Pi\oplus\Pi')$$

$$(V\oplus V')_i = V_i\oplus V_i' \quad , \quad (\Pi\oplus\Pi')(a) = \underset{\begin{pmatrix}\Pi(a) & 0\\ 0 & \Pi'(a)\end{pmatrix}}{\underset{\|}{\Pi(a)\oplus\Pi'(a)}}$$

Repr. is indecomposable if it cannot be decomposed in a direct sum of nonzero representations. (not isom. to such a direct sum)

Thm. Any repr. is uniquely a direct sum of indecomposable representations. (Krull-Schmidt thm.)

Thm. Q is of finite type ⟺ it has finitely many indecomp. representations, and the number of them is independent on the field F and of the orientation.

Ex. For E_8, there are 120 repr.

* th. Gabriel: only if: hwk

if: hard (will use technology of root systems)

Ex. 1

$A_1 : \quad \bullet \quad \Rightarrow \quad V \cong F^n$

Indec. repr. is F infin. by $\bullet 1$ $\Rightarrow$ 1 rep.

Ex. 2

$A_2 : \quad \bullet \longrightarrow \bullet \quad \Rightarrow \quad \overset{V}{\bullet} \xrightarrow{A} \overset{W}{\bullet}$

$Y = \operatorname{Ker} A$

Y' = complement of Y in V

$$\Rightarrow \quad V \xrightarrow{A} W = \begin{array}{c} Y \longrightarrow 0 \\ \oplus \\ \boxed{Y' \xrightarrow{A} W} \end{array} \rightsquigarrow \text{multiple of} \quad F \longrightarrow 0$$

$\wr$

denote by

$1 \longrightarrow 0$

$\underbrace{1 \to 1 \oplus \cdots \oplus 1 \to 0}_{\dim Y \text{ times}}$

$\|$

$\dim Y \cdot 1 \longrightarrow 0$

split away for A.

Now:

$W' = \operatorname{Im} A$ and choose a complement Z of W' in W

$$\begin{array}{c} Y \longrightarrow 0 \\ \oplus \\ Y' \xrightarrow{A} W' \\ \oplus \\ 0 \longrightarrow Z \end{array} \qquad \begin{array}{c} \\ \\ \Rightarrow \quad Y' \cong W' \\ \\ \rightsquigarrow \quad \dim Z \cdot 0 \longrightarrow 1 \end{array}$$

Pick a basis $e_1, \dots, e_n$ of V'. then

$Ae_1, \dots, Ae_n$ basis of W', so

$$V' \xrightarrow{A} W' = \mathbb{C}e_1 \xrightarrow{A} \mathbb{C}Ae_1 \oplus \cdots \oplus \mathbb{C}e_n \xrightarrow{A} \mathbb{C}Ae_n$$

$$1 \longrightarrow 1 \oplus \cdots \oplus 1 \longrightarrow 1$$

<u>Indec:</u> $1 \longrightarrow 0$

$0 \longrightarrow 1$

$1 \longrightarrow 1$

If A is $m \times n$ of rank p, then rep.

Π_A of A_2 is

$$\Pi_A = (n-p)\cdot[1 \longrightarrow 0] \oplus (m-p)[0 \longrightarrow 1] \oplus p \cdot [1 \longrightarrow 1]$$

<u>More gen:</u> A_n has $\frac{n(n+1)}{2}$ indec.