Remind:

$\boxed{2}$ $\mathbb{A}_1$ $\overset{\circ}{1}$

$\mathbb{A}_2$ $\circ \xrightarrow{A} \circ$ $\qquad \overset{\circ}{0} \longrightarrow \overset{\circ}{1}$

$\overset{\circ}{1} \longrightarrow \overset{\circ}{0}$

$\overset{\circ}{1} \longrightarrow \overset{\circ}{1}$

$\mathbb{A}_3$ $\underset{V_1}{\circ} \xrightarrow{A_1} \underset{W}{\circ} \xleftarrow{A_2} \underset{V_2}{\circ}$

step 1 split away $\ker A_1$, $\ker A_2$.
$\qquad\qquad\qquad\quad \overset{\shortparallel}{K_1} \qquad \overset{\shortparallel}{K_2}$

$V_1 \longrightarrow W \longleftarrow V_2$

$K_1 \longrightarrow 0 \longleftarrow 0 \oplus 0 \longrightarrow 0 \longleftarrow K_2 \oplus$

$V_1' \hookrightarrow W \hookleftarrow V_2'$

where $V_1'$ is the complement of $K_1$ in $V_1$, and $V_2'$ is the complement of $K_2$ in $V_2$.

we focus in step 2.

pair of subspaces problem

Indec. representations of $\cdot \longrightarrow \cdot \longleftarrow \cdot$

$1 \longrightarrow 0 \longleftarrow 0$
$0 \longrightarrow 0 \longleftarrow 1$ ] st. 1

$1 \longrightarrow 1 \longleftarrow 1$ ] st. 2

$0 \longrightarrow 1 \longleftarrow 0$
$1 \longrightarrow 1 \longleftarrow 0$
$0 \longrightarrow 1 \longleftarrow 1$ ] st. 3

$6 = \frac{3(3+1)}{2}$

$V_1 \hookrightarrow W \hookleftarrow V_2$

Step 2 : Split away intersection $Y = V_1 \cap V_2$

We come $V_1' = V_1$ $V_2' = V_2$ (larger ?)

~~$V_1'$ is complement of $Y$ in $V_1$~~

~~$V_2'$ is complement of $Y$ in $V_2$.~~

stuck

Let $W'$ be a complement of $Y$ in $W$

Define $V_1' = W' \cap V_1$, $V_2' = W' \cap V_2$.

Now, it works:

$$V_1 \hookrightarrow W \hookleftarrow V_2 = \underbrace{Y \longrightarrow Y \longleftarrow Y}_{\text{multiple of } 1 \to 1 \leftarrow 1} \oplus \underbrace{V_1' \hookrightarrow W' \longleftarrow V_2'}_{\text{step 3}}, \quad (V_1' \cap V_2' = 0)$$

Step 3

$V_1 \hookrightarrow W \hookleftarrow V_2$ $\quad V_1 \cap V_2 = 0$

$V_1 \oplus V_2 \subset W$

Split away $W / V_1 \oplus V_2$.

Choose ~~the~~ a complement $Y$ of $V_1 \oplus V_2$ in $W$

$$V_1 \longrightarrow W \longleftarrow V_2 = (V_1 \longrightarrow V_1 \oplus V_2 \longleftarrow V_2) \oplus$$

$$\underbrace{0 \longrightarrow Y \longleftarrow 0}_{\text{multiple of } 0 \to 1 \leftarrow 0}$$

$V_1 \to V_1 \oplus V_2 \leftarrow V_2 =$

$= V_1 \to V_1 \leftarrow 0 \quad \oplus \quad 0 \to V_2 \leftarrow V_2$

↳ multiple of $1 \to 1 \leftarrow 0$

↳ mult. of $0 \to 1 \leftarrow 1$

Indec. rep. of

$\bullet \to \bullet \to \bullet$

$1 \to 0 \to 0$ ] st. 1
$0 \to 0 \to 1$
$1 \to 1 \to 0$ ] st. 2
$0 \to 1 \to 1$ ] st. 3
$1 \to 1 \to 1$ ] st. 4
$0 \to 1 \to 0$

We have also 6 dec.

$A_3 \quad \underset{V}{\bullet} \xrightarrow{A} \underset{W}{\bullet} \xrightarrow{B} \underset{Y}{\bullet}$

Step 1 • Split away $\ker A$

$\to$ we get $1 - 0 - 0$

$V \to W \to Y = V' \to W \to Y \oplus$

$\ker A \to 0 \to 0$ ↳ $(1 \to 0 \to 0)$

• Split away $\operatorname{coker} B$

Get to: $V \xhookrightarrow{A \text{ injective}} W \xrightarrow{B} \!\!\!\to Y$

Step 2 • Split away $\ker(B \circ A) = X$

let $X'$ be a complement of $X$ in $V$

$W'$ - a complement of $A(X)$ in $W$ which contains $A(X')$

$V \to W \to Y = X \xrightarrow{\sim} A(X) \to 0 \oplus \left( X' \xhookrightarrow{A} W' \xrightarrow{B} \!\!\!\to Y \right)$

(because $A$ is injective)

(multiple of $1 \to 1 \to 0$)

$\ker(B \circ A) = \{0\}$

Step 3 Split away Coker $(B \circ A)$

splits away a multiple of $0 \longrightarrow 1 \longrightarrow 1$

Get to a situation when

$V \overset{A}{\hookrightarrow} W \overset{B}{\twoheadrightarrow} Y$, the map $B \circ A$ has trivial kernel, and trivial cokernel, so is an isomorphism.

Step 4 $V \hookrightarrow W \twoheadrightarrow Y = \underbrace{V \xrightarrow[\sim]{A} A(V) \xrightarrow[\sim]{B} Y}$ (mult. of $1 \to 1 \to 1$)

$\oplus$ $0 \to W' \to 0$ (mult. of $0 \to 1 \to 0$)

[complement of $A(V)$ in $W$]

Observe that the number of decompositions and the order of the numbers in this decompositions are exactly the same. This is always going to happen (we will prove it). But the decomposition are not exactly the same because of the direction of the arrows.

=

$D_4$
$$\begin{tikzcd} V_1 \arrow[r, "\Delta_1"] & V & V_3 \arrow[l, "\Delta_3"'] \\ & V_2 \arrow[u, "\Delta_2"] & \end{tikzcd}$$

Spoiler!

There will be 12 dec.

Step 1: Split away $\ker \Delta_1$, $\ker \Delta_2$, $\ker \Delta_3$.

$$\boxed{\begin{tikzcd} 1 \arrow[r] & 0 & 0 \arrow[l] \\ & 0 \arrow[u] & \end{tikzcd}}_1 \qquad \boxed{\begin{tikzcd} 0 \arrow[r] & 0 & 0 \arrow[l] \\ & 1 \arrow[u] & \end{tikzcd}}_2$$

$$\boxed{\begin{tikzcd} 0 \arrow[r] & 0 & 1 \arrow[l] \\ & 0 \arrow[u] & \end{tikzcd}}_3$$

Get to
$$\begin{tikzcd} V_1 \arrow[r, hook, "\Delta_1"] & V & V_3 \arrow[l, hook', "\Delta_3"'] \\ & V_2 \arrow[u, hook, "\Delta_2"] & \end{tikzcd}$$

triple of subspaces problem

invariants:
- $\dim V$ 1
- $\dim V_i$ 3
- $\dim V_i \cap V_j$ 3
- $\dim V_1 \cap V_2 \cap V_3$ 1

$\dim (V_1 + V_2 + V_3) =$ cannot be determined by the invariants above.

Step 2 Split away $V_1 \cap V_2 \cap V_3 = Y$

Choose a complement $V'$ of $Y$ in $V$,

and $V_i' = V' \cap V_i$
$$\begin{tikzcd} V_1 \arrow[r] & V & V_3 \arrow[l] \\ & V_2 \arrow[u] & \end{tikzcd} =$$

$$= \underbrace{\begin{tikzcd} Y \arrow[r] & Y & Y \arrow[l] \\ & Y \arrow[u] & \end{tikzcd}}_{1} \oplus \begin{tikzcd} V_1' \arrow[r] & V' & V_3' \arrow[l] \\ & V_2' \arrow[u] & \end{tikzcd}$$

$V_1' \cap V_2' \cap V_3' = 0$

$$\boxed{\begin{tikzcd} 1 \arrow[r] & 1 & 1 \arrow[l] \\ & 1 \arrow[u] & \end{tikzcd}}_4$$

## Step 3: Split away complement of $V_1+V_2+V_3$ in $V$

Pick a complement $Y$ like this.

$$\begin{tikzcd} V_1 \arrow[r] & V & V_3 \arrow[l] \\ & V_2 \arrow[u] & \end{tikzcd} = \begin{tikzcd} V_1 \arrow[r] & V_1+V_2+V_3 & V_3 \arrow[l] \\ & V_2 \arrow[u] & \end{tikzcd} \oplus$$

Whole space

$$\oplus \begin{tikzcd} 0 \arrow[r] & Y & 0 \arrow[l] \\ & 0 \arrow[u] & \end{tikzcd}$$

mult of $\boxed{\begin{tikzcd} 0 \arrow[r] & 1 & 0 \arrow[l] \\ & 0 \arrow[u] & S \end{tikzcd}}$

We have:

$$\begin{tikzcd} V_1 \arrow[r] & V & V_3 \arrow[l] \\ & V_2 \arrow[u] & \end{tikzcd}$$

where $V_1+V_2+V_3 = V$

$V_1 \cap V_2 \cap V_3 = 0$

## Step 4

Let $Y = V_1 \cap V_2$. Choose a complement $V'$ of $Y$ in $V$

How do we choose the complement?

$V_1' = V' \cap V_1$, $V_2' = V' \cap V_2$

So that $V_3 \subset V'$, we can do that because $V_3 \cap Y = 0$
$\to \exists\, V'$ compl. of $Y$ in $V$ that contains $V_3$.

$$\begin{tikzcd} V_1 \arrow[r] & V & V_3 \arrow[l] \\ & V_2 \arrow[u] & \end{tikzcd} = \begin{tikzcd} V_1' \arrow[r] & V' & V_3 \arrow[l] \\ & V_2' \arrow[u] & \end{tikzcd} \oplus$$

$V_1' \cap V_2' \cap V_3 = 0$

$$\oplus \begin{tikzcd} Y \arrow[r] & Y & 0 \arrow[l] \\ & Y \arrow[u] & \end{tikzcd} \sim \boxed{\begin{tikzcd} 1 \arrow[r] & 1 & 0 \arrow[l] \\ & 1 \arrow[u] & \end{tikzcd}}$$

In a similar way, we get:

$$\begin{tikzcd} 1 \arrow[r] & 1 & 1 \arrow[l] \\ & 0 \arrow[u] & \end{tikzcd} \quad 7$$

$$\begin{tikzcd} 0 \arrow[r] & 1 & 1 \arrow[l] \\ & 1 \arrow[u] & \end{tikzcd} \quad 8$$

Get to:

$$\begin{tikzcd} V_1 \arrow[r] & V & V_3 \arrow[l] \\ & V_2 \arrow[u] & \end{tikzcd}$$

$V_1 + V_2 + V_3 = V$

$V_1 \cap V_2 = 0$

$V_1 \cap V_3 = 0$

$V_2 \cap V_3 = 0$

Step 5 Suppose $V_1 \not\subseteq V_2 \oplus V_3$

Let $Y = V_1 \cap (V_2 \oplus V_3)$, $V_1'$ is a complement of $Y$ in $V_1$.

$V_1' \cap (V_2 \oplus V_3) = 0 \longrightarrow$ There exists a complement $V'$ of $V_1'$ in $V$ containing $V_2 \oplus V_3$. So we choose this complement

$$\begin{tikzcd} V_1 \arrow[r, hook] & V & V_3 \arrow[l, hook'] \\ & V_2 \arrow[u, hook] & \end{tikzcd} = \begin{tikzcd} V_1' \arrow[r, "\sim"] & V_1' & 0 \arrow[l] \\ & 0 \arrow[u] & \end{tikzcd} \oplus$$

$$\begin{tikzcd} 1 \arrow[r] & 1 & 0 \arrow[l] \\ & 0 \arrow[u] & \end{tikzcd} \quad 9$$

$$\oplus \quad \begin{tikzcd} Y \arrow[r, hook] & V' & V_3 \arrow[l, hook'] \\ & V_2 \arrow[u, hook] & \end{tikzcd}$$

Similar way:

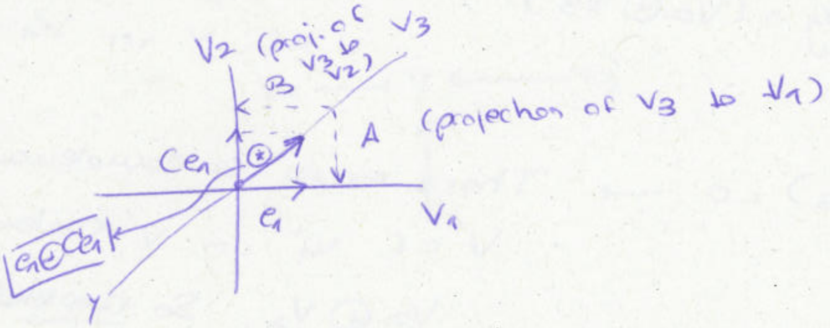
$$\begin{tikzcd} 0 \arrow[r] & 1 & 0 \arrow[l] \\ & 1 \arrow[u] & \end{tikzcd} \qquad 10$$

$$\begin{tikzcd} 0 \arrow[r] & 1 & 1 \arrow[l] \\ & 0 \arrow[u] & \end{tikzcd} \qquad 11$$

Get to:
$$\begin{cases} V_1 \cap V_2 = V_2 \cap V_3 = V_3 \cap V_1 = 0, \\ V_1 + V_2 + V_3 = V, \\ V_1 \subset V_2 \oplus V_3, \quad V_2 \subset V_1 \oplus V_3, \quad V_3 \subset V_1 \oplus V_2 \end{cases}$$

$$\Longrightarrow V_1 \oplus V_2 = V_2 \oplus V_3 = V_3 \oplus V_1 = V$$

$$\Longrightarrow \begin{cases} \dim(V_1) = \dim(V_2) = \dim(V_3) = n, \\ \dim(V) = 2n. \end{cases}$$




$A, B$ isom. $(\dim(V_1) = \dim(V_2) = \dim(V_3))$

$C : V_1 \longrightarrow V_2$, $\quad C = B \circ A^{-1}$

$V_3$ - graph of $C$. This means that if $V_1$ has basis $e_1, \ldots, e_n$; then
$$\begin{cases} V_1 = \mathbb{C}e_1 \oplus \cdots \oplus \mathbb{C}e_n \\ V_2 = \mathbb{C}Ce_1 \oplus \cdots \oplus \mathbb{C}Ce_n \\ V_3 = \mathbb{C}(e_1 \oplus Ce_1) \oplus \cdots \oplus \mathbb{C}(e_n \oplus Ce_n) \end{cases}$$

$\Longrightarrow$ We get:

$$\begin{tikzcd} 1 \arrow[r] & 2 & 1 \arrow[l] \\ & 1 \arrow[u] & \end{tikzcd} \qquad 12$$