If A is $m \times n$ of rank p then rep. π_A of A_2 is

$$\pi_A = (n-p)\cdot[1 \longrightarrow 0] \oplus (m-p)[0 \longrightarrow 1] \oplus p(1 \longrightarrow 1]$$

More general: A_n has $\frac{n(n+1)}{2}$ indec.

② A_1 $\overset{\cdot}{1}$

A_2 $\cdot \longrightarrow \cdot$ $\quad {}_0\cdot \longrightarrow \cdot_1$ (1-dimensional v.s.)

${}_1\cdot \longrightarrow \cdot_0$ (0-dim. v.s.)

${}_0\cdot \longrightarrow \cdot_0$

A_3 $\underset{V_1}{\cdot} \xrightarrow{A_1} \underset{W}{\cdot} \xleftarrow{A_2} \underset{V_2}{\cdot}$

<u>Step 1</u> Split away $\underset{K_1}{\overset{\shortparallel}{\operatorname{Ker} A_1}}$, $\underset{K_2}{\overset{\shortparallel}{\operatorname{Ker} A_2}}$

$V_1 \longrightarrow W \longleftarrow V_2 =$

$= K_1 \longrightarrow 0 \longleftarrow 0 \oplus 0 \longrightarrow 0 \longleftarrow K_2 \oplus$

$\oplus\ V_1' \longrightarrow W \longleftarrow V_2'$ pair of subspaces problem

V_1' - complement of K_1 in V_1

V_2' - " " K_2 in V_2

$$V_1 \hookrightarrow W \hookleftarrow V_2$$

Step 2: Split away intersection $Y = V_1 \cap V_2$

Let W' be a complement of Y in W

Define $V_1' = W' \cap V_1$, $V_2' = W' \cap V_2$

$$V_1 \hookrightarrow W \hookleftarrow V_2 = Y \longrightarrow Y \longleftarrow Y \oplus$$

multiple of $1 \rightarrow 1 \leftarrow 1$

$$V_1' \hookrightarrow W' \hookleftarrow V_2'$$

$V_1' \cap V_2' = 0$!

Step 3: $V_1 \hookrightarrow W \hookleftarrow V_2$ $\quad V_1 \cap V_2 = 0$

$V_1 \oplus V_2 \subset W$

Split away $W / V_1 \oplus V_2$

Choose a complement Y of $V_1 \oplus V_2$ in W

$$V_1 \longrightarrow W \longleftarrow V_2 =$$

$$= V_1 \longrightarrow V_1 \oplus V_2 \longleftarrow V_2 \oplus 0 \longrightarrow Y \longleftarrow 0$$

(multiple of $0 \rightarrow 1 \leftarrow 0$)

$$V_1 \longrightarrow V_1 \oplus V_2 \longleftarrow V_2 =$$

$$= V_1 \longrightarrow V_1 \longleftarrow 0 \oplus 0 \longrightarrow V_2 \longleftarrow V_2$$

(multiple of $1 \rightarrow 1 \leftarrow 0$) (multiple of $0 \rightarrow 1 \leftarrow 1$)

Indec. rep. of $\cdot \rightarrow \cdot \leftarrow \cdot$

$1 \rightarrow 0 \leftarrow 0$

$0 \rightarrow 0 \leftarrow 1$

$1 \rightarrow 1 \leftarrow 1$

$0 \rightarrow 1 \leftarrow 0$ $\quad 6 = \frac{3(3+1)}{2}$

$0 \rightarrow 1 \leftarrow 1$

$1 \rightarrow 1 \leftarrow 0$

$A_3 \quad \underset{V}{\cdot} \xrightarrow{A} \underset{W}{\cdot} \xrightarrow{B} \underset{Y}{\cdot}$

Step 1 Split away Ker A:

$V \rightarrow W \rightarrow Y = V' \rightarrow W \rightarrow Y \oplus \mathrm{Ker}\, A \rightarrow 0 \rightarrow 0$

Split away Coker B

Get to: $V \underset{A}{\hookrightarrow} W \underset{B}{\twoheadrightarrow} Y$

<u>Step 2</u> Split away $\mathrm{Ker}\,(B \cdot A) = X$

X' complement of X in V

W' a complement of $A(X)$ in W which contains $A(X')$ (Remember A is injective)

$V \rightarrow W \rightarrow Y =$

$= X \xrightarrow{\simeq} A(X) \rightarrow 0 \oplus \underline{X' \hookrightarrow W' \twoheadrightarrow Y}$

(multiple of $1 \rightarrow 1 \rightarrow 0$) $\quad \mathrm{Ker}\, B \cdot A = 0$

<u>Step 3</u>: Split away Coker $B\circ A$

Splits away a multiple of $0 \to 1 \to 1$

Get to a situation when $V \xhookrightarrow{A} W \xrightarrow{B} Y$ the map $B\circ A$ is an isom

<u>Step 4</u>: $V \hookrightarrow W \twoheadrightarrow Y =$

$$= V \xrightarrow[\sim]{A} A \xrightarrow[\sim]{B} Y \oplus 0 \to W' \to 0$$

multiple of $1 \to 1 \to 1$ multiple of $0 \to 1 \to 0$

W' complement of $A(V)$ in W

$A_3 \quad \cdot \longrightarrow \cdot \longrightarrow \cdot$

$1 \to 0 \to 0$

$0 \to 0 \to 1$

$1 \to 1 \to 0$

$0 \to 1 \to 1$

$0 \to 1 \to 0$

$1 \to 1 \to 1$

D_4

$$\begin{tikzcd} \underset{V_1}{\cdot} \arrow[r, "A_1"] & \overset{V}{\cdot} & \cdot\, V_3 \arrow[l, "A_3"'] \\ & \cdot\, V_2 \arrow[u, "A_2"'] & \end{tikzcd}$$

Spoiler: $\underline{12}$

Step 1: Split away $\ker A_1, \ker A_2, \ker A_3$

Get to:
$$\begin{tikzcd} V_1 \arrow[r, hook, "A_1"'] & V & V_3 \arrow[l, hook', "A_3"] \\ & V_2 \arrow[u, hook, "A_2"] & \end{tikzcd}$$

triple of subspaces problem

invariants: $\underset{1\ n^\circ}{\dim V}$, $\underset{3n^\circ}{\dim V_i}$, $\underset{3n^\circ}{\dim V_i \cap V_j}$, $\underset{1\ n^\circ}{\dim V_1 \cap V_2 \cap V_3}$

$\underset{1\ n^\circ}{\dim(V_1 + V_2 + V_3)} \Rightarrow 9$ invariants

Step 2: Split away $V_1 \cap V_2 \cap V_3 = Y$

Choose a compl. V' of Y in V and

$V_i' = V' \cap V_i$
$$\begin{tikzcd} V_1 \arrow[r] & V & V_3 \arrow[l] \\ & V_2 \arrow[u] & \end{tikzcd} =$$

$$= \begin{tikzcd} Y \arrow[r] & Y & Y \arrow[l] \\ & Y \arrow[u] & \end{tikzcd} \oplus \begin{tikzcd} V_1' \arrow[r] & V' & V_3' \arrow[l] \\ & V_2' \arrow[u] & \end{tikzcd}$$

$(V_1' \cap V_2' \cap V_3' = 0)$

Step 3: Split away complement of $V_1 + V_2 + V_3$ in V. Pick a complement Y like this

$$\begin{tikzcd} V_1 \arrow[r] & V & V_3 \arrow[l] \\ & V_2 \arrow[u] & \end{tikzcd} =$$

whole space

$$= \begin{tikzcd} V_1 \arrow[r] & V_1+V_2+V_3 & V_3 \arrow[l] \\ & V_2 \arrow[u] & \end{tikzcd} \oplus \begin{tikzcd} 0 \arrow[r] & Y & 0 \arrow[l] \\ & 0 \arrow[u] & \end{tikzcd}$$

$$\left(\text{multiple of } \begin{tikzcd} 0 \arrow[r] & 1 & 0 \arrow[l] \\ & 0 \arrow[u] & \end{tikzcd} \right)$$

$$\begin{tikzcd} V_1 \arrow[r] & V & V_3 \arrow[l] \\ & V_2 \arrow[u] & \end{tikzcd} \qquad \begin{aligned} V_1 + V_2 + V_3 &= V \\ V_1 \cap V_2 \cap V_3 &= 0 \end{aligned}$$

$Y = V_1 \cap V_2$ Choose a compl. V' of Y in V

$V_1' = V' \cap V_1$, $V_2' = V' \cap V_2$

Choose V' so that $V_3 \subset V'$

$Y \cap V_3 = 0$

$$\begin{tikzcd} V_1 \arrow[r] & V & V_3 \arrow[l] \\ & V_2 \arrow[u] & \end{tikzcd} = \begin{tikzcd} V_1' \arrow[r] & V' & V_3 \arrow[l] \\ & V_2' \arrow[u] & \end{tikzcd} \oplus \begin{tikzcd} Y \arrow[r] & Y & 0 \arrow[l] \\ & Y \arrow[u] & \end{tikzcd}$$

Get to: $\begin{tikzcd} V_1 \arrow[r] & V & V_3 \arrow[l] \\ & V_2 \arrow[u] & \end{tikzcd}$

$V_1 + V_2 + V_3 = V$, $V_1 \cap V_2 = 0$, $V_1 \cap V_3 = 0$, $V_2 \cap V_3 = 0$

$\to V_1 + V_2 + V_3 = V$

<u>Step 5</u>: Suppose $V_1 \not\subseteq V_2 \oplus V_3$

$Y = V_1 \cap (V_2 \oplus V_3)$.

V_1' complement of Y in V_1

$V_1' \cap (V_2 \oplus V_3) = 0 \to \exists V'$ compl. of V_1' in V cont. $V_2 \oplus V_3$

$$\begin{tikzcd} V_1 \arrow[r, hook] & V & V_3 \arrow[l, hook'] \\ & V_2 \arrow[u, hook] & \end{tikzcd} =$$

$$= \begin{tikzcd} V_1' \arrow[r, "\sim"] & V_1' & 0 \arrow[l] \\ & 0 \arrow[u] & \end{tikzcd} \oplus \begin{tikzcd} Y \arrow[r, hook] & V' & V_3 \arrow[l, hook'] \\ & V_2 \arrow[u, hook] & \end{tikzcd}$$

mult. of $\begin{tikzcd} 1 \arrow[r] & 1 & 0 \arrow[l] \\ & 0 \arrow[u] & \end{tikzcd}$

Get to: $V_1 \cap V_2 = V_2 \cap V_3 = V_3 \cap V_1 = 0$,

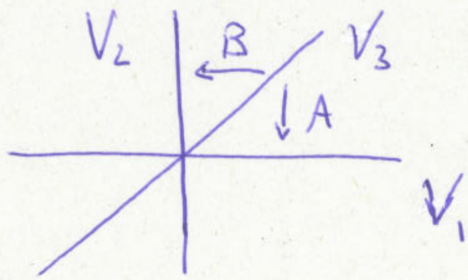
$V_1 + V_2 + V_3 = V \quad V_1 \subset V_2 \oplus V_3 \quad V_2 \subset V_1 \oplus V_3$

$V_3 \subset V_1 \oplus V_2$

$V_1 \oplus V_2 = V_2 \oplus V_3 = V_3 \oplus V_1 = V$

$\dim V_1 = \dim V_2 = \dim V_3 = n, \quad \dim V = 2n$

A, B isom.



$C : V_1 \longrightarrow V_2 \quad C = B \cdot A^{-1} \quad V_3$-graph of C

If V_1 has basis $e_1, \dots, e_n$

Then: $V_1 = \mathbb{C}e_1 \oplus \cdots \oplus \mathbb{C}e_n$

$V_2 = \mathbb{C} \cdot Ce_1 \oplus \cdots \oplus \mathbb{C} \cdot Ce_n$

$V_3 = \mathbb{C}(e_1 \oplus Ce_1) \oplus \cdots \oplus \mathbb{C}(e_n \oplus Ce_n)$

$$\begin{tikzcd} 1 \arrow[r] & 0 & 0 \arrow[l] \\ & 0 \arrow[u] & \end{tikzcd} \qquad \begin{tikzcd} 0 \arrow[r] & 0 & 0 \arrow[l] \\ & 1 \arrow[u] & \end{tikzcd} \qquad \begin{tikzcd} 0 \arrow[r] & 0 & 1 \arrow[l] \\ & 0 \arrow[u] & \end{tikzcd}$$

$$\begin{tikzcd} 1 \arrow[r] & 1 & 1 \arrow[l] \\ & 1 \arrow[u] & \end{tikzcd} \qquad \begin{tikzcd} 0 \arrow[r] & 1 & 0 \arrow[l] \\ & 0 \arrow[u] & \end{tikzcd} \qquad \begin{tikzcd} 1 \arrow[r] & 1 & 0 \arrow[l] \\ & 1 \arrow[u] & \end{tikzcd}$$

$$\begin{tikzcd} 1 \arrow[r] & 1 & 1 \arrow[l] \\ & 0 \arrow[u] & \end{tikzcd} \qquad \begin{tikzcd} 0 \arrow[r] & 1 & 1 \arrow[l] \\ & 1 \arrow[u] & \end{tikzcd} \qquad \begin{tikzcd} 1 \arrow[r] & 1 & 0 \arrow[l] \\ & 0 \arrow[u] & \end{tikzcd}$$

$$\begin{tikzcd} 0 \arrow[r] & 1 & 0 \arrow[l] \\ & 1 \arrow[u] & \end{tikzcd} \qquad \begin{tikzcd} 0 \arrow[r] & 1 & 1 \arrow[l] \\ & 0 \arrow[u] & \end{tikzcd} \qquad \begin{tikzcd} 1 \arrow[r] & 2 & 1 \arrow[l] \\ & 1 \arrow[u] & \end{tikzcd}$$