$$\rightarrow d(F_i^+ V) - d(V) = -B_Q(d(V), \alpha_i)\alpha_i$$

$$\rightarrow d(F_i^+ V) = d(V) - B_Q(d(V), \alpha_i)\alpha_i = s_i d(V)$$

$\boxed{5}$ Coxeter elements

Q - Dynkin diagram vertices $1, 2, \dots, n$

$W = \langle s_i \rangle \qquad s_i(x) = x - B(x, \alpha_i)\alpha_i$

$\llcorner$ Weyl group $\qquad B = B_Q$

Def: The Coxeter element $c \in W$ is:

$$c = s_1 s_2 \cdots s_n$$

Lemma: Let $\beta = \sum_{i=1}^{n} k_i \alpha_i$ with $k_i \geq 0 \ \forall i$, $\beta \neq 0$. Then $\exists N$ such that $c^N \beta$ has at least one negative coefficient.

Proof: $c: \mathbb{R}^n \longrightarrow \mathbb{R}^n$ I claim that c does
$\qquad \mathbb{C}^n \longrightarrow \mathbb{C}^n$

not have eigenvalue 1.

Indeed, suppose $cv = v \Leftrightarrow s_1 s_2 \cdots s_n v = v$

$\rightarrow s_2 \cdots s_n v = s_1 v$

first coord. of $s_2 \cdots s_n v$ is the same as that of v. Also other coord., since it is also $s_1 v \Rightarrow s_1 v = v \Rightarrow$

$\Rightarrow S_2 S_3 \cdots S_n v = v \to S_3 \cdots S_n v = S_2 v$

this is so again v, so $S_2 v = v, \ldots$

so we get that $s_i v = v$ for all $i = 1, \ldots, n$

$\to B(v, \alpha_i) = 0 \ \forall i \to v = 0$, as desired.

Also, since W is finite, there exists M such that $c^M = 1 \Rightarrow c^M - 1 = 0 \Rightarrow$

$\to \underbrace{(c-1)}(c^{M-1} + c^{M-2} + \cdots + c + 1) = 0$

invertible since 1 is not an eigenvalue of c.

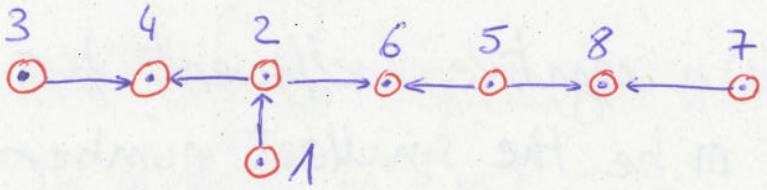
$\to c^{M-1} + c^{M-2} + \cdots + c + 1 = 0 \Rightarrow$

$\to c^{M-1}\beta + c^{M-2}\beta + \cdots + c\beta + \beta = 0$

Since β has coord. ≥ 0 and one of them is > 0, some coord. of $c^N\beta$ is < 0 for some N.

Proof of Gabriel's theorem

Let Q be a quiver of type ADE (so we fix an orientation). Have partial order on vertices: $i < j$ if can go from i to j. Can complete it to total order. Label vertices $1, 2, \ldots, n$ according to this total order. If can go from i to j then $i < j$



(Removing sinks)

Let V be a rep of Q, indecomposable

Define a sequence

$$V^{(0)} = V,\ V^{(1)} = F_n^+ V^{(0)},\ V^{(2)} = F_{n-1}^+ V^{(1)} = F_{n-1}^+ F_n^+ V^{(0)},$$

$$\ldots,\ \underline{V^{(n)} = F_1^+ \ldots F_n^+ V^{(0)}},$$

same orientation as Q since each edge got reversed twice.

$$, V^{(n+1)} = F_n^+ F_1^+ \ldots F_n^+ V^{(0)}, \ldots$$

<u>Prop. 1</u>: There exists $m \in \mathbb{N}$ such that $d(V^{(m)}) = \alpha_p$ — a simple root

<u>Proof</u>: $V^{(i+1)} = F_k^+ V^{(i)}$. If $V^{(i)}$ is surjective at k then $d(V^{(i+1)}) = s_k d(V^{(i)})$

So if $V^{(0)}, V^{(1)}, \ldots, V^{(k)}$ are surjective at the appropiate vertices then

$$d(V^{(i)}) = \underbrace{\cdots\ s_{n-1} s_n}_{i \text{ factors}} d(V^{(0)})$$

By lemma, it cannot contiue to infinity, since $s_1 \cdots s_n = c$ a Coxeter element and

$c^N d(V^{(0)})$ has a negative coefficient for some N. Let m be the smallest number such that $V^{(m)}$ is not surjective at the appropriate vertex, call it p. Since $V^{(m)}$ indec., we get that $d(V^{(m)}) = \alpha_p$

$$\begin{tikzcd} & & p \arrow[dl] \\ 0 \arrow[r] & 1 & 0 \arrow[l] \\ & 0 \arrow[u] & \end{tikzcd}$$

<u>Corollary</u>: $d(V)$ is a positive root.

<u>Proof</u>: Pick m as in prop. 1 such that $V^{(0)}, \dots, V^{(m)}$ are indec., $d(V^{(m)}) = \alpha_p$ and $V^{(m+1)} = 0$. We get $\underset{\alpha_p}{\underset{\|}{d(V^{(m)})}} = s_r \cdots s_n d(V)$

$\Rightarrow s_n s_{n-1} \cdots s_r \alpha_p = d(V)$

$\Rightarrow d(V)$ is a root (positive)

<u>Corollary 2</u>: If V and V' are two indec. reps with dimension vector $d(V) = d(V') = \alpha$, then $V \cong V'$.

<u>Proof</u>: Let m be as in prop. 1 for V, and m' be as in prop. 1 for V'. Assume that $m \leq m'$.

$$d(V^{(m)}) \overset{\alpha_p}{\overset{\shortparallel}{=}} s_r \cdots s_{n-1} s_n \alpha = d(V'^{(m)})$$

(so in fact $m' = m$)

$$V^{(m)} = V'^{(m)} = \mathbb{C}_{(p)} \qquad \begin{tikzcd} 0 \arrow[r] & \overset{p}{1} & 0 \arrow[l] \\ & 0 \arrow[u] & \end{tikzcd}$$

$$F_n^- F_{n-1}^- \cdots F_r^- \mathbb{C}_{(p)} \cong \begin{cases} F_n^- \cdots F_r^- F_r^+ \cdots F_n^+ V \cong V \\ F_n^- \cdots F_r^- F_r^+ \cdots F_n^+ V' \cong V' \end{cases}$$

<u>Corollary 3</u>: For every positive root α there is an indec. rep. of Q of dimension α.

<u>Proof</u>: Look at sequence $\alpha, s_n\alpha, s_{n-1}s_n\alpha, \ldots,$ and consider the last term which is still a positive root. $s_r \cdots s_{n-1} s_n \alpha = \beta$ - positive root but $s_{r-1}\beta$ negative $\to \beta = \alpha_{r-1}$, so

$\alpha = s_n s_{n-1} \cdots s_r \alpha_{r-1}$

Now take and apply F^-: $\mathbb{C}_{(r-1)}$

$$F_n^- F_{n-1}^- \cdots F_r^- \mathbb{C}_{(r-1)} \qquad \begin{tikzcd} & & r-1 \arrow[dl] \\ {} \arrow[r] & 1 & {} \arrow[l] \\ & {} \arrow[u] & \end{tikzcd}$$

This is an indec rep., of dimension α

$\boxtimes$

Gabriel

$$\begin{tikzcd} \overset{1}{\underset{\color{red}1}{1}} \arrow[r] & \overset{2}{\underset{\color{red}4}{\cdot}} & \overset{3}{\underset{\color{red}3}{1}} \arrow[l] \\ & \overset{\color{red}2}{1} \arrow[u] & \end{tikzcd}$$

labels dim

$\alpha = \alpha_1 + \alpha_2 + \alpha_3 + 2\alpha_4$

$$\begin{tikzcd} 0 \arrow[r] & \mathbb{C} & 0 \arrow[l] \\ & 0 \arrow[u] & \end{tikzcd} \xleftrightarrow{F_1^-} \begin{tikzcd} \mathbb{C} & \mathbb{C} \arrow[l] & 0 \arrow[l] \\ & 0 \arrow[u] & \end{tikzcd} \xleftrightarrow{F_2^-}$$

$$\begin{tikzcd} \mathbb{C} & \mathbb{C} \arrow[l] \arrow[d] & 0 \arrow[l] \\ & \mathbb{C} & \end{tikzcd} \xleftrightarrow{F_3^-} \begin{tikzcd} \mathbb{C} & \mathbb{C} \arrow[l] \arrow[r] \arrow[d] & \mathbb{C} \\ & \mathbb{C} & \end{tikzcd} \longleftrightarrow$$

$$\begin{tikzcd} \mathbb{C} \arrow[r] & \mathbb{C}^3/\mathrm{diag} & \mathbb{C} \arrow[l] \\ & \mathbb{C} \arrow[u] & \end{tikzcd} \qquad \cong \mathbb{C}^2$$

$\mathbb{C} \longrightarrow \mathbb{C}^3 \qquad x \longrightarrow (x, x, x)$