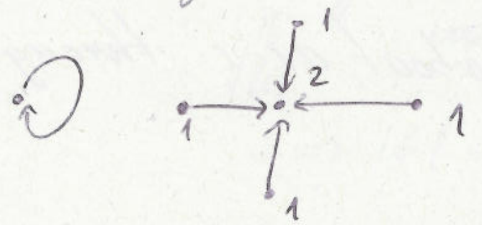
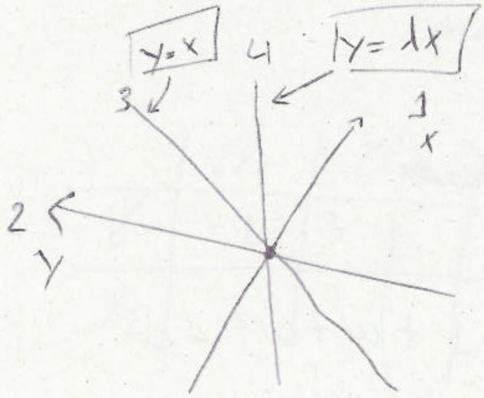
Pavel Etingof



Are of infinite type



, $\lambda \in \mathbb{C}$ is cross ratio

4 lines in 2-dimensional plane.

Roots: Fix a Quiver Q of type A_n, D_n, E_6, E_7, E_8
(Dynkin ~~..~~ simply laced diagrams)

Cartan Matrix , $A_Q = 2\,\mathrm{Id} - R_Q = (a_{ij})$

R_Q - Adjacency Matrix of Q $\quad (R_Q)_{ij} = \begin{cases} 1, & i-j \\ 0, & \text{else} \end{cases}$

$$a_{ij} = \begin{cases} 2, & i=j \\ -1, & i-j \\ 0, & \text{else} \end{cases}$$

E.g. For A_4, it's
$$\begin{pmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{pmatrix}$$

$B_Q(x,y) = x^t A_Q y$ - bilinear form in $\mathbb{R}^n$ (n number of vertices)

Lemma 1) B_Q is positive definite

2) If $x \in \mathbb{Z}^n$, then $B_Q(x,x) \in 2\mathbb{Z}$ (even)

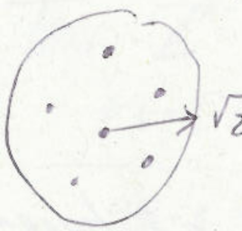
Proof: 1) HWK + Peter's lectures.

2) $B_Q(x,x) = 2\sum_{i=1}^{n} x_i^2 - 2\sum_{\substack{i<j \\ i-j}} x_i x_j$, even. $\square$

Def: A root is a vector $x \in \mathbb{Z}^n$ s.t. $B_Q(x,x) = 2$.

Lemma 2. There are finitely many roots.

Proof: They form a discrete set inside a ball.



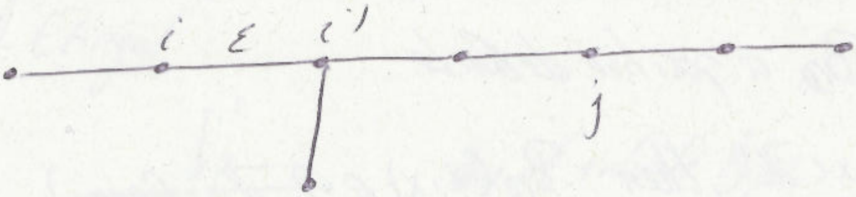
Ex.: $\alpha_i = (0,0,\dots,\underset{\uparrow i}{1},0,\dots,0)$

$\{\alpha_1,\dots,\alpha_n\}$, $B_Q(\alpha_i,\alpha_i) = 2$, so α_i are roots

- α_i are called simple roots.

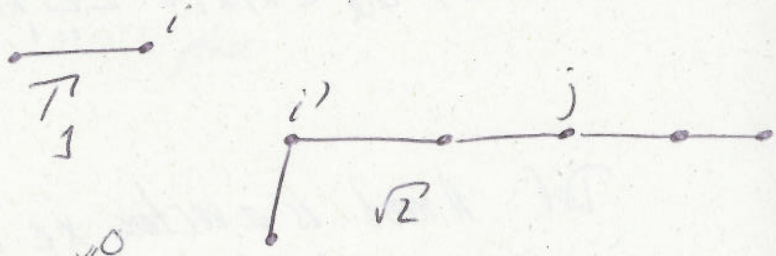
Lemma 3. If $\alpha = \sum_{i=1}^{n} k_i \alpha_i$ is a root, then either $k_i \geq 0$ for all i or $k_i \leq 0$ for all i. ($k_i \in \mathbb{Z}$)

Proof: Assume the contrary, $k_i > 0$, but $k_j < 0$ ($i \neq j$).



Draw a path from i to j, let i' be next to i on this path, .

~~Rename:~~ Remove ε:

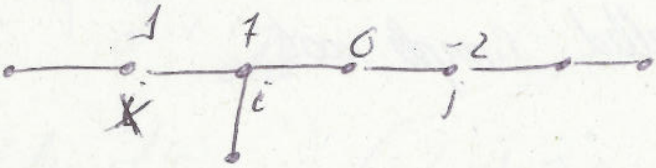


$$\overset{\neq 0}{\beta} = \sum_{m \in T_1} k_i \alpha_m , \qquad \overset{\neq 0}{\gamma} = \sum_{m \in T_2} k_m \alpha_m$$

$B_Q(\beta, \beta) \geq 2$, $B_Q(\gamma, \gamma) \geq 2$.

$$B_Q(\beta, \gamma) = - k_i k_{i'} \geq 0$$

WLOG, assume that $k_s = 0$ for all s between i and j.



Because either $k_{i'} = 0$ or $i' = j$ then $k_{i'} < 0$ and $k_i > 0$.

$$\Rightarrow B_Q(\beta+\gamma, \beta+\gamma) = B_Q(\beta, \beta) + B_Q(\gamma, \gamma) + 2 B_Q(\beta, \gamma) \geq 4 \Rightarrow \Leftarrow \quad \alpha = \beta + \gamma \text{ is a root } \square.$$

Def: A positive root is a root $\alpha = \sum_i k_i \alpha_i$ s.t $k_i \geq 0$

So any root is positive or negative

Ex.: Q of type A_{N-1}

$\mathbb{Z}^{N-1} \subset \mathbb{Z}^N$ - set of $x = [x_1, \dots, x_N]$ s.t. $\sum_i x_i = 0$

$(\ , \ )$ - usual inner product on $\mathbb{Z}^N$.

$(x, y) = \sum_i x_i y_i$

This $\mathbb{Z}^{N-1}$ has basis

$\alpha_1 = [1, -1, 0, \dots, 0]$, $\alpha_2 = [0, 1, -1, \dots, 0]$, $\alpha_{N-1} = [0, \dots, 0, 1, -1]$

$(\alpha_i, \alpha_i) = 2$, $\underset{i \neq j}{(\alpha_i, \alpha_j)} = \begin{cases} 0 & , |i-j| \geq 2 \\ -1 & , |i-j| = 1 \end{cases}$

Exactly A_Q.

Roots: $\sum_i x_i = 0$, $\sum_i x_i^2 = 2$

So $\alpha = [0, \dots, \underset{i}{1}, \dots, \underset{j}{-1}, \dots, 0]$ or $[0, \dots, -1, 1, \dots, 0]$

Positive — Negative

$\alpha = \alpha_i + \alpha_{i+1} + \dots + \alpha_{j-1}$

So we see that for A_{N-1}, have $\frac{N(N-1)}{2}$ pos. roots

Fact: (HwK) $D_N = N(N-1)$

E_6 36

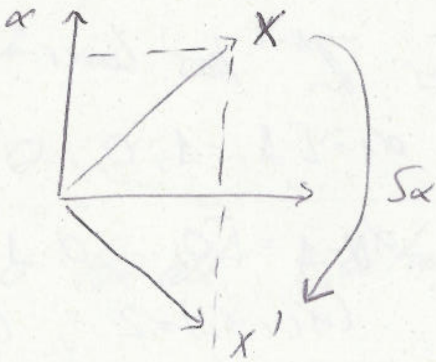
E_7 63

E_8 120

Root $\alpha \rightsquigarrow S_\alpha : \mathbb{R}^n \longrightarrow \mathbb{R}^n$

$$S_\alpha(X) = X - B_Q(X,\alpha)\alpha$$

Reflection



$S_\alpha(\alpha) = -\alpha$

$S_\alpha(X) = X$ if $X \perp \alpha$

$(B_Q(X,\alpha) = 0)$

$S_i \overset{def}{=} S_{\alpha_i}$, Simple Reflections

It's clear that Reflections preserve B_Q:

$B_Q(x,y)$ (exercise)

Def: The Weyl Group $W \subset O(B_Q) \subseteq GL_n(\mathbb{R})$

$O(B_Q) = O_n(\mathbb{R})$ Orthogonal Group is the subgroup generated by the s_i.

lemma 4. W maps the root lattice $\mathbb{Z}^n = \mathbb{Z}\alpha_1 \oplus \dots \oplus \mathbb{Z}\alpha_n$ to itself, and maps roots to roots, and it is a finite group.

Proof: Clearly $s_i(\mathbb{Z}^n) \subset \mathbb{Z}^n$
and s_i maps roots to roots since preserve B_Q.
let $w \in W$ then w is determined by $w(\alpha_i)$, but there are roots, so there are finitely many choices for $w(\alpha_i)$.
In fact, $|W| \leq (\#\text{of roots})^n$. $\square$

<u>Ultimate Version of Gabriel Theorem:</u>

Suppose that we have a representation V of quiver Q which has spaces $V_1, \dots, V_n$ at vertices $1, \dots, n$.

• <u>Def</u> The vector $(\dim V_1, \dots, \dim V_n)$ is called the dimension vector of V and is denoted $d(V)$.

<u>Gabriel's Theorem:</u> ("if" direction) let Q be a quiver of type A_n, D_n, E_6, E_7, E_8. Then the dimension vector of any indecomposable representation of Q is a positive root, and for every positive root there is exactly one indecomp. representation.

So # indec. rep = # positive roots $< \infty$, e.g.

for A_n get $\frac{n(n+1)}{2}$

$\alpha = \sum_i k_i \alpha_i \longrightarrow (k_1, \dots, k_n)$ 1, 3, 6,

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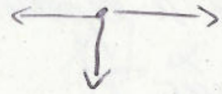
<u>Reflection Functors</u>: Q - quiver

Def.: vertex $i \in Q$ is a sink if all edges conn. to i go to i.

i is a source if all edges conn. to i go out of i.



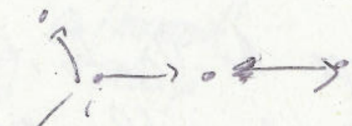
 neither source or sink.

$\forall i \in Q$, let $\bar{Q}_i$ - quiver obtained from Q by reversing the arrows at i.

Q 

Def: Let $i \in Q$ be a sink.
Then define the reflection functor

$F_i^+ : \mathrm{Rep}\, Q \longrightarrow \mathrm{Rep}\, \bar{Q}_i$.

$\bar{Q}_i$ 

(machine making repr. of $\bar{Q}_i$ from repr. of Q).

$V \in Rep\, Q \quad , V = (V_j)$

$F_i^+(V)_j = V_j$

$j \neq i$

$F_i^+(V)_i \overset{def}{=} Ker(\bigoplus_{i-j} A_{ij}) \qquad \forall j - i$

$A_{ij} : V_j \to V_i$

$\bigoplus_{j-i} A_{ij} : \bigoplus_{j-i} V_j \longrightarrow V_i$

F_i^+ does not change maps sitting on edges which don't connect to i.

For $K - i$, the corresponding map $V'_i \longrightarrow V_K$ is just the projection to V_j from $\bigoplus_{K^i} V_K$

Def: $i \in Q$ source. $F_i^- : Rep\, Q \longrightarrow Rep\, \bar{Q}_i$

$$\begin{array}{ccccc} & & V_i & & \\ \overset{V_j}{\bullet} & \longleftarrow & \bullet & \longrightarrow & \bullet \\ & & \downarrow & & \\ & & \bullet & & \end{array}$$

$F_i^-(V)_K = V_K$ if $K \neq i$

$F_i^-(V)_i \overset{def}{=} Coker(\bigoplus_{K-i} A_{Ki}) =$

$= \bigoplus_{i-K} V_K / Im(\bigoplus_{K^i} A_{Ki})$

$\bigoplus_{K-i} A_{Ki} : V_i \longrightarrow \bigoplus_{K-i} V_K$

F_i^- does not touch maps on edges not connecting to i.

For $i - j$, the correspondente map is the natural $V_i' \longleftarrow V_K$ map

$V_K \longrightarrow \bigoplus V_j \longrightarrow \bigoplus_{j-i} V_j / Im(\bigoplus A_{ji})$

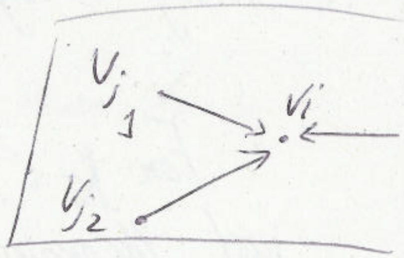
Prop 1. Let V be an indec. repn. of Q.

1) If i is a ~~simple~~ sink, then either

$\dim V_i = 1$, $\dim V_j = 0$, $j \neq i$

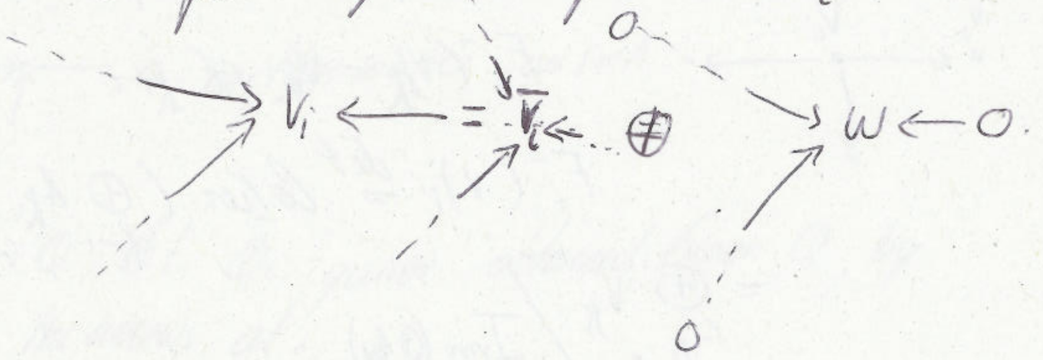
~~For~~ or $\varphi = \bigoplus_{i-j} A_{ij} : \bigoplus_{i-j} V_j \longrightarrow V_i$ is surjective

2) If i is a source, then either $\dim V_i = 1$, $\dim V_j = 0$, $j \neq i$



or $\psi \overset{def}{=} \bigoplus_{i-j} A_{ji} : V_i \longrightarrow \bigoplus_{j-i} V_j$ is injective

Proof: 1) Split away the comp. of $\mathrm{Im}\,\varphi$ in V_i



indec. $\Rightarrow$ One of them is ~~zero~~ 0.

$\bar{V}_i = \mathrm{Im}\,\varphi$

2-nd is 0 $\Rightarrow$ φ is surjective

W - complement

1st is 0

$V_i = \bar{V}_i \oplus W$.

$\square$

Proposition 2. Let V be a repr. of Q.

1) If $\varphi : \bigoplus_{j-i} V_j \longrightarrow V_i$ is surjective (i is a sink),

then $F_i^- F_i^+ V = V$

2) If $\psi : V_i \longrightarrow \bigoplus_{j-i} V_j$ is injective (i is a source)

then $F_i^+ F_i^- V = V$.

Proof: Suppose $\varphi : \bigoplus_{j-i} V_j \longrightarrow V_i$ is surj, and $K = \operatorname{Ker}\varphi$.

Apply F_i^+ : Replace V_i with K and maps with $K \longrightarrow V_j$. Now, if we apply F_i^- at vertex i, we will get $\bigoplus_j V_j / K = \operatorname{Ker}\varphi = \operatorname{Im}\varphi = V_i$

Exercise: Check we have also the same maps

$\square$

Prop. 3 Let V be an indec. rep n. of Q.

1) If i a sink, then $F_i^+ V$ is either indecomposable or zero.

2) If i is a source, then $F_i^- V$ is either indec. or zero.

$$V = \begin{tikzcd} 0 \arrow[r] & 1 & 0 \arrow[l] \\ & 0 \arrow[u] & \end{tikzcd}$$

$F_i^+ V = 0$

Proof: Prove 1) By Prop. 1, either φ is surjective or $\dim V_i = 1$, $\dim V_j = 0$, $j \neq i$.

In the latter case, $F_i^+ V = 0$, so we can assume φ is surjective. Assume $F_i^+ V = X \oplus Y$

Our job is to show that X or Y is 0.

$F_i^+ V$ is injective at i. (ψ is injective)

$\psi : \operatorname{Ker}\varphi \hookrightarrow \bigoplus_{j-i} V_j$

$\Rightarrow X$ and $Y \overset{\neq 0}{}$ ~~are injective at $i \Rightarrow$ by Prop 2~~ we want to get a contradiction.

$$F_i^+ \underbrace{F_i^-}\, X = X, \quad F_i^+ \underbrace{F_i^-}\, Y = Y \Rightarrow$$

Are nonzero

$$V = F_i^- F_i^+ V \longrightarrow \text{holds by Prop 2}$$

$$\| $$

$$F_i^- X \oplus F_i^- Y \text{ , decomposable!}$$

$$\rightarrow F_i^- F_i^+ V = F_i^- (X \oplus Y) = F_i^- X \oplus F_i^- Y.$$

Prop 4. Let V be a repr. of Q.

1) i a sink, φ is surjective at $i \Rightarrow$
$$d(F_i^+ V) = s_i\, d(V)$$

2) i a source, ψ is injective at $i \Rightarrow$
$$d(F_i^- V) = s_i\, d(V)$$

Proof: 1) $\left(d(F_i^+ V) - d(V)\right)_i = \dim \operatorname{Ker}\varphi - \dim V_i$

By Rank-Nullity Theorem, $\left(\sum_{j-i} \dim V_j - \dim V_i\right) - \dim V_i$

$= \sum_{j-i} \dim V_j - 2 \dim V_i$

$$(d(F_i^+ V) - d(V))_j = 0$$

$\Rightarrow$

$$d(F_i^+ V) - d(V) = - B_Q(d(V), \alpha_i)\alpha_i$$

$$\Rightarrow d(F_i^+ V) = d(V) - B_Q(d(V), \alpha_i)\alpha_i =$$

$= s_i\, d(V)$. $\square$.