1. Find a one dimensional subspace of traceless two-by-two real matrices whose image under the exponential map is compact.

Solution and comment. The image under exp of $\begin{pmatrix} 0 & t \\ -t & 0 \end{pmatrix}$ is the compact circle group

$$\mathrm{SO}(2) = \begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix}.$$

The same considerations show the exponentional of any subalgebra of skew symmetric matrices will generate a connected closed subgroup of the compact group $\mathrm{SO}(n)$. This will be relevant ater when we discuss how the classification of semimsimple Lie algebras is related to the classificationof compact connected Lie groups.

2. Verify that the image of traceless real matrices under the exponential map consists of matrices with determinant one. Is the exponential map surjecive onto matrices of determinant one?

Solution and comment. The first verification is straightforward. The matrix $\begin{pmatrix} -1 & 1 \\ 0 & -1 \end{pmatrix}$ is not in the image of the exponential map. This illustrates one of the (many!) subtleties of the Lie correspondence which we formulated and discussed in the first lecture.

3. In the formal expression

$$Z = \log(\exp(X)\exp(Y))$$

we verified in the lecture that

$$Z = X + Y + \frac{1}{2}[X,Y] + \cdots .$$

Write the cubic terms in $\cdots$ explicitly in terms of iterated brackets of X and Y.

Solution and comment. This painful manipulation illustrates why the explicit formula (rather than abstract statement) of the Baker-Campbell-Hausdorf result resisted many attempts. Here is the general formula due to Dynkin:

$$\sum \frac{(-1)^{k-1}}{k} \frac{1}{(i_1+j_i)+\cdots+(i_k+j_k)} \frac{[X^{(i_1)}Y^{(j_1)}\cdots X^{(i_k)}Y^{(j_k)}]}{i_1!j_1!\cdots i_k!j_k!}$$

where the sum is over all $2k$ tuples, for each $k \geq 1$, of nonnegative integers $(i_1,\ldots,i_k,j_1,\ldots,j_k)$ satisfying $i_r + j_r \geq 1$. The bracket notation is defined as

$$[A_1, A_2, \ldots, A_k] := [A_1, [A_2, \ldots [A_{k-1}, A_k], \ldots]],$$

and

$$[X^{(i_1)}Y^{(j_1)}\cdots X^{(i_k)}Y^{(j_k)}]$$

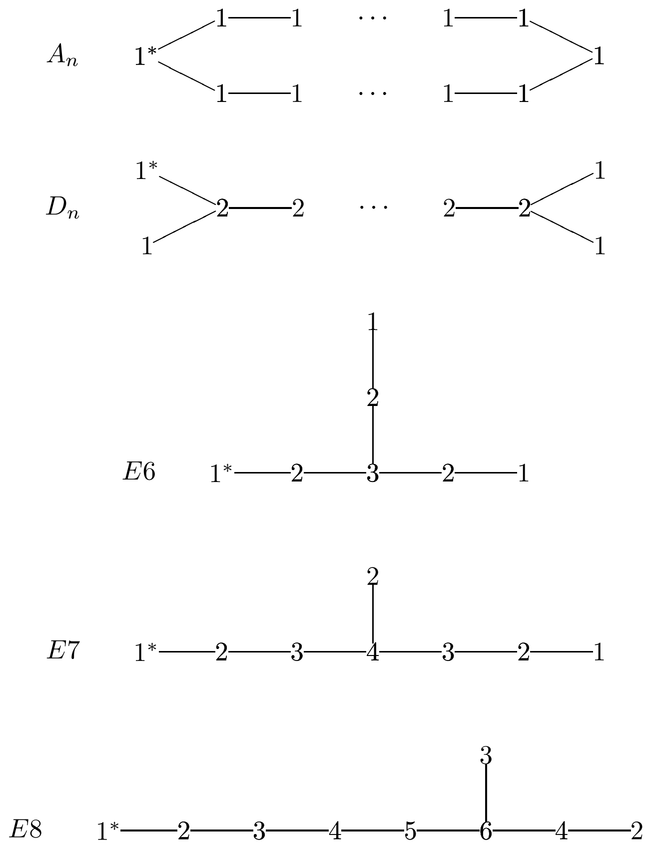
means take X i_1 times, then Y j_1 times, and so on inside the above bracket notation.

Fortunately the precise formula is really not all that important! Instead the remarkable fact is the Lie algebra bracket controls the groups multiplication near the identify (and this is really what is needed for the Lie correspondence for linear groups).

4. Classify all finite, connected, undirected graphs with the following properties:

(a) There are no self-loops in the graph.
(b) Between any two vertices there is at most one edge.
(c) Every vertex is labeled with a positive integer.
(d) The label at any vertex v is equal to one-half the sum of the labels of the vertices connected to v.
(e) There is a distinguished vertex labeled 1^*.

Solution and comment. This is perhaps the simplest place where the ADE classfication emerges. In addition to the two infinite families A_n and D_n, there are just three exceptional graphs. (This is a complete surprise, just as in Gabriel's Theorem, since with such simple rules one would expect either no excetions at all or else a very long list of them.) We record them here:



5. **(In connection with Sommers' lectures and Etingof's exercises)** Consider the so-called binary tetrahedral group

$$G = \langle x, y, z \,| x^2 = y^3 = z^3 = xyz \rangle.$$

Write down the character table of G and let V be the unique two dimensional irreducible representation of G such that $V \otimes V$ contains the trivial representation. Build a graph with vertices indexed by the irreducible representation of G labeled by the dimension of their irreducible representations. (The trivial representation provides a distinguished vertex labeled 1.) Using the character table, connect two vertices corresponding to irreducible π_1 and π_2 by an (undirected edge) if π_1 appears in $\pi_2 \otimes V$. (Why is appropriate that the edge is undirected?) Have you seen this graph before? What are the natural questions to ask at this point?

6. Beginning with the abstract constuction of the root system of Type D4 given in the lecture (rather than a google search) compute the Weyl group of Type D4. Generalize to D_n. Compare with the abstract Coxeter presentation discussed in the lectures.

7. Attach a matrix to each of the graphs appearing in Exercise 4 whose rows and columns are indexed by the unstarred vertices of the graphs, whose diagonal entries are 2, and whose off diagonal ij entry is -1 if i and j are connected and zero otherwise. (In other words, this is the matrix of the nondegenerate bilinear form used to construct the root system.) Compute the determinant in all cases. How is the result related to the automorphisms of the graphs?

8. Note that E6 has a diagram automorphism of order 2 fixing the special vertex. Perform the folding construction of the root system of E6 to obtain a four-dimensional root system (no longer simply laced) of Type F4. Verify that the abstract construction is equivalent to the following useful concrete realization. Let V be $\mathbb{R}^4$ with the usual inner product and orthonormal basis $e_1, \ldots, e_4$. Let R be the union of $\{\pm e_i \mid 1 \leq i \leq 4\}$, $\{\pm(e_i \pm e_j) \mid 1 \leq i \neq j \leq 4\}$, and $\{\pm \frac{1}{2}(e_1 \pm e_2 \pm e_3 \pm e_4)\}$. (Probably you should start by writing down a base for R.)