1. Find a one dimensional subspace of traceless two-by-two real matrices whose image under the exponential map is compact.

Solution and comment. The image under exp of $\begin{pmatrix} 0 & t \\ -t & 0 \end{pmatrix}$ is the compact circle group

$$\mathrm{SO}(2) = \begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix}.$$

The same considerations show the exponentional of any subalgebra of skew symmetric matrices will generate a connected closed subgroup of the compact group $\mathrm{SO}(n)$. This will be relevant ater when we discuss how the classification of semimsimple Lie algebras is related to the classificationof compact connected Lie groups.

2. Verify that the image of traceless real matrices under the exponential map consists of matrices with determinant one. Is the exponential map surjecive onto matrices of determinant one?

Solution and comment. The first verification is straightforward. The matrix $\begin{pmatrix} -1 & 1 \\ 0 & -1 \end{pmatrix}$ is not in the image of the exponential map. This illustrates one of the (many!) subtleties of the Lie correspondence which we formulated and discussed in the first lecture.

3. In the formal expression

$$Z = \log(\exp(X)\exp(Y))$$

we verified in the lecture that

$$Z = X + Y + \frac{1}{2}[X, Y] + \cdots .$$

Write the cubic terms in $\cdots$ explicitly in terms of iterated brackets of X and Y.

Solution and comment. This painful manipulation illustrates why the explicit formula (rather than abstract statement) of the Baker-Campbell-Hausdorf result resisted many attempts. Here is the general formula due to Dynkin:

$$\sum \frac{(-1)^{k-1}}{k} \frac{1}{(i_1 + j_i) + \cdots + (i_k + j_k)} \frac{[X^{(i_1)}Y^{(j_1)} \cdots X^{(i_k)}Y^{(j_k)}]}{i_1!j_1! \cdots i_k!j_k!}$$

where the sum is over all $2k$ tuples, for each $k \geq 1$, of nonnegative integers $(i_1, \ldots, i_k, j_1, \ldots, j_k)$ satisfying $i_r + j_r \geq 1$. The bracket notation is defined as

$$[A_1, A_2, \ldots, A_k] := [A_1, [A_2, \ldots [A_{k-1}, A_k], \ldots]],$$

and

$$[X^{(i_1)}Y^{(j_1)} \cdots X^{(i_k)}Y^{(j_k)}]$$

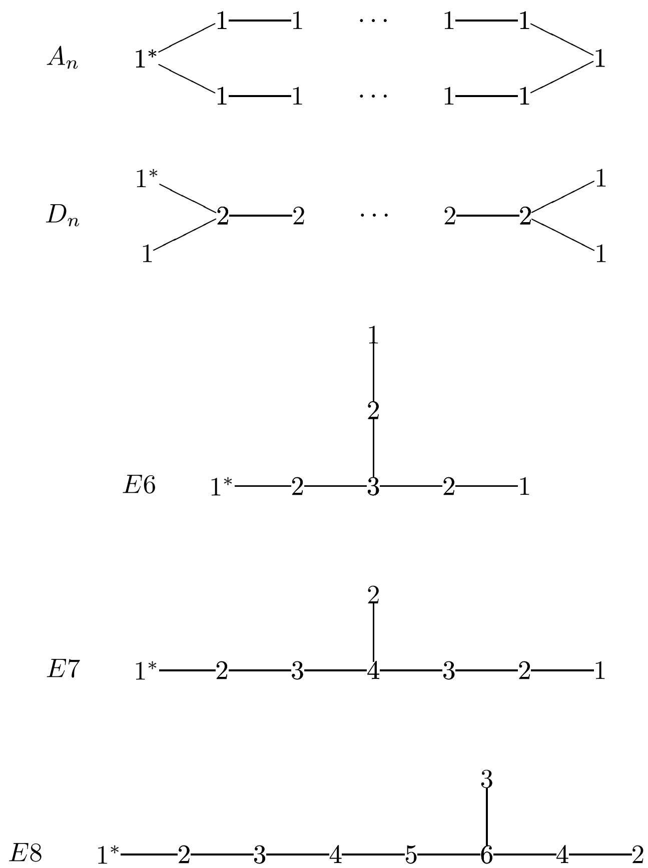
means take X i_1 times, then Y j_1 times, and so on inside the above bracket notation.

Fortunately the precise formula is really not all that important! Instead the remarkable fact is the Lie algebra bracket controls the groups multiplication near the identify (and this is really what is needed for the Lie correspondence for linear groups).

4. Classify all finite, connected, undirected graphs with the following properties:

(a) There are no self-loops in the graph.
(b) Between any two vertices there is at most one edge.
(c) Every vertex is labeled with a positive integer.
(d) The label at any vertex v is equal to one-half the sum of the labels of the vertices connected to v.
(e) There is a distinguished vertex labeled 1^*.

Solution and comment. This is perhaps the simplest place where the ADE classfication emerges. In addition to the two infinite families A_n and D_n, there are just three exceptional graphs. (This is a complete surprise, just as in Gabriel's Theorem, since with such simple rules one would expect either no excetions at all or else a very long list of them.) We record them here:



5. **(The McKay correspondece)** Consider the so-called binary tetrahedral group

$$G = \langle x, y, z \mid x^2 = y^3 = z^3 = xyz \rangle.$$

Write down the character table of G, and let V be the unique two dimensional irreducible representation of G such that the class function χ_V^2 contains the character of the trivial representation when expressed in the basis of irreducible characters (as discussed in Sommers' lectures). Build a graph with vertices indexed by the irreducible representation of G labeled by their dimension. (The trivial representation provides a distinguished vertex labeled 1.) Using the character table, connect two vertices corresponding to irreducibles π_1 and π_2 by an (undirected edge) if the character of π_1 appears in the decomposition of the product of the characters of π_2 and V in the basis of irreducible characters. (Why is appropriate that the edge is undirected?) Have you seen this graph before? What are the natural questions to ask at this point?

Comments and hints. The reason for the terminology "binary tetrahedral" is as follows. One can quickly verify that xyz is a central element of order 2, and that the quotient of G by the normal subgroup $\{e, xyz\}$ is the alternating group A_4 of symmetries of the tetrahedron.

Write p for the projection from G to A_4 via the quotient map. Note that if $\pi : G \to \mathrm{GL(U)}$ is an irreducible represenation of A_4, then $\pi \circ p$ is an irreducible representation of G. Since Sommers constructed the character table of A_4 (consisting of one three dimensional irreducible and three one dimensional representations) at the end of his fourth lecture, we immediately obtain 4 irreducible representations of G. The discussion above shows that G has order equal to twice the order of A_4, hence 24 elements. In the expression of 24 as a sum of squares of irreducible representations, we have already accounted for $3^2 + 1^2 + 1^2 + 1^2$ from the irreducible representations of A_4. One can then provide an argument (based on counting congugacy classes, for example) that we are missing three 2 dimensional representations of G. Once of them is the V mentioned in the exercise. The characters of the other two are obtained by multiplying the character χ_V of V by the two nontrivial one-dimensional characters.

We can play the same game for the symmetry groups of the other four platonic solids (cube, octahedron, dodecahedron, and icosahedron). Since the cube and octahedron have the same symmetry groups, as do the dodecahedron and icosahedron, we are reduced to studying the binary cube group and binary icosahedron group,

$$G_{\text{cube}} = \langle x, y, z \,|x^2 = y^3 = z^4 = xyz\rangle$$
$$G_{\text{icos}} = \langle x, y, z \,|x^2 = y^3 = z^5 = xyz\rangle$$

the corresponding covers of the symmetry groups of the cube and icosahedron. Write down the character tables for these groups. Once again there is a two-dimensional V characterized by the requirement that χ_V^2 contains the trivial character in its expression into irreducible characters. One can build a graph in the identical way and discover a surprising coincidence (first observed by John McKay).

Where does the double cover come from? The symmetry group of solids in $\mathbb{R}^3$ naturally live in SO(3). The Lie group SO(3) has fundamental group $\mathbb{Z}/2$ and so admits a simply conneted double cover, SU(2) in this case. The binary symmetry groups above are the preimages of the symmetry groups inside SU(2). This endows them with a two dimensional representation V (the two-by-two matrices of SU(2)), and it is this V we use above.

To complete the McKay correspondence, we also have to look at the binary dihedral group (arising from symmetries of the n-gon)

$$G_{\text{n}-\text{gon}} = \langle x, y, z \,|x^2 = y^3 = z^n = xyz\rangle,$$

and the cyclic group $\mathbb{Z}/n$ (arising from symmetries of the n-gon where we distinguish the top face from the bottom in $\mathbb{R}^3$). These two infinite families correspond to the diagrams of type D and A, respectively.

One tiny comment: we could have replaced products of characters everywhere in this discussion with tensor products of representations (but we did not discuss tensor products in these lectures).

6. Beginning with the abstract constuction of the root system of Type D4 given in the lecture (rather than a google search) compute the Weyl group of Type D4. Generalize to D_n. Compare with the abstract Coxeter presentation discussed in the lectures.

7. Attach a matrix to each of the graphs appearing in Exercise 4 whose rows and columns are indexed by the unstarred vertices of the graphs, whose diagonal entries are 2, and whose off diagonal ij entry is -1 if i and j are connected and zero otherwise. (In other words, this is the matrix, sometimes called the Cartan matrix, of the nondegenerate bilinear form used to construct the root system. Compare with Exercise 5.3 in Etingof's notes.) Compute the determinant in all cases. How is the result related to the automorphisms of the graphs?

8. Note that E6 has a diagram automorphism of order 2 fixing the special vertex. Perform the folding construction of the root system of E6 to obtain a four-dimensional root system (no longer simply laced) of Type F4. Verify that the abstract construction is equivalent to the following useful concrete realization. Let V be $\mathbb{R}^4$ with the usual inner product and orthonormal basis $e_1, \dots, e_4$. Let R be the union of $\{\pm e_i \mid 1 \leq i \leq 4\}$, $\{\pm(e_i \pm e_j) \mid 1 \leq i \neq j \leq 4\}$, and $\{\pm\frac{1}{2}(e_1 \pm e_2 \pm e_3 \pm e_4)\}$. (Probably you should start by writing down a base for R.)

9. In the lectures, we carefully constructed the roots systems of Type A1 and A2 from $\mathfrak{sl}(n, \mathbb{C})$ for $n = 2, 3$. Repeat the construction carefully for arbitrary n. (This isn't especially difficult, but it will clarify many of the general details, particularly the inner product derived from the Killing form.)