One ideia to understand $\mathfrak{g}$ (with complicated brackets) is to try to "restrict" to a subspace $\mathfrak{G} \subseteq \mathfrak{g}$ such that $\forall x, y \in \mathfrak{G}, [x, y] = 0$.
Notice: A subspace satisfying the above condition is a subalgebra and is called abelian.

Notation: Given $X \in \mathfrak{G}$ define

$$ad(X) \in End_{\mathbb{C}}(\mathfrak{g}) = \mathfrak{M}(n, \mathbb{C})$$

(where $n = dim(\mathfrak{g})$) via $ad(X)Y = [X, Y]$.

We can check (it need Jacobi identity) that forall $X, Y \in \mathfrak{g}$

$$ad \underbrace{[X, Y]}_{brakets\, in\, \mathfrak{g}} = \underbrace{[ad(X), ad(Y)]}_{brakets\, in\, \mathfrak{M}(\mathfrak{n}, \mathbb{C})}$$

Might hope that for $X \in \mathfrak{G}$, $ad(X)$ is a diagonalizable endomorphism of $\mathfrak{g}$.

Example 1 Consider $\mathfrak{g} := \mathfrak{M}(2, \mathbb{C})$. Is it simple? Consider the multiples of Id,

$$\{Z_t := \begin{bmatrix} t & 0 \\ 0 & t \end{bmatrix} | t \in \mathbb{C}\}$$

Since $[Z_t, X] = 0, \forall X$, then, it's a non trivial ideal. Then $\mathfrak{g}$ is not simple.

Example 2 Take

$$\mathfrak{g} := \left\{ \begin{bmatrix} h & x \\ y & -h \end{bmatrix} | h, x, y \in \mathbb{C} \right\}$$

Take $\mathfrak{G} = \mathbb{C}H$ where $H = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$. Set $X = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$, and $Y = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$.

So, $\mathfrak{g} := \mathfrak{G} \oplus \mathbb{C}X \oplus \mathbb{C}Y$.
Look at $ad(H) : \mathfrak{g} \to \mathfrak{g}$, and compute

$$ad(H)X = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} - \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = 2X$$

Same calculation shows that $ad(H)Y = -2Y$.

Define $\alpha \in \mathfrak{G}^* = Hom_{\mathbb{C}}(\mathfrak{G}, \mathbb{C})$, by $\alpha(H) = 2$.
Define,

$$\mathfrak{g}^{\alpha} = \{A \in \mathfrak{g} |\, ad(Z)A = \alpha(Z)A, \forall Z \in \mathfrak{G}\} = \mathbb{C}X$$

$$\mathfrak{g}^{\alpha} = \{A \in \mathfrak{g} | ad(Z)A = -\alpha(Z)A\} = \mathbb{C}Y$$

Set $R := \{+\alpha, -\alpha\} \subseteq \mathfrak{G}^*$.

$$R: \; -\alpha \longleftarrow \text{A} \longrightarrow \alpha$$

Then

$$\mathfrak{g} = \mathfrak{G} \oplus (\bigoplus_{\beta \in R} \mathfrak{g}^{\beta})$$

Small technical wrinkle If $\mathfrak{G}$ is a maximal abelian subalgebra of $\mathfrak{g}$ and $X \in \mathfrak{G}$, then $ad(X) : \mathfrak{g} \to \mathfrak{g}$ need not to be diagonalizable.

Bad Example Take $\mathfrak{G} = \mathbb{C} \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \in \mathfrak{sl}(2, \mathbb{C})$. We have

$$ad \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}^3 = 0$$

Definition A *Cartan subalgebra* $\mathfrak{G}$ of $\mathfrak{g}$ is a subalgebra such that

1. $\forall x, y \in \mathfrak{G}, \exists N$ such that $ad(X)^n Y = 0$ (nilpotent)
2. If $Z \in \mathfrak{g}$ such that $ad(Z)\mathfrak{G} \subseteq \mathfrak{G}$ then $Z \in \mathfrak{G}$ (self-normalizing)

Example Consider $\mathfrak{G} = \mathbb{C} \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \in \mathfrak{sl}(2, \mathbb{C})$, violates condition 2, because

$$\left(ad \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \right) \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = 2 \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \in \mathfrak{G}$$

Theorem *(Cartan)* A Cartan subalgebra $\mathfrak{G}$ is maximally abelian and for all $H \in \mathfrak{G}$, $ad(H)$ is a diagonalizable endomorphism of $\mathfrak{g}$.

Let $R \subseteq \mathfrak{G}^*$ consists of those $\alpha : \mathfrak{G} \to \mathbb{C}$ non-zero, such that $\mathfrak{g}^{\alpha} := \{x \in \mathfrak{g} | ad(H)X = \alpha(H)X\}$ (joint eigenspace of $\{ad(H) : H \in \mathfrak{G}\}$ with joint eigenvalue α). Since all of $ad(H)$ are diagonalizable $s|$ commute

$$\mathfrak{g} = \mathfrak{g}^0 \oplus (\bigoplus_{\alpha in R} \mathfrak{g}^{\alpha})$$

Since $\mathfrak{G}$ is maximally abelian,

$$\mathfrak{g} = \mathfrak{G} \oplus (\bigoplus_{\alpha in R} \mathfrak{g}^{\alpha})$$

Example Consider

$$\mathfrak{G} = \{ \begin{bmatrix} h_1 & 0 & 0 \\ 0 & h_2 & 0 \\ 0 & 0 & h_3 \end{bmatrix} : h_1 + h_2 + h_3 = 0\}$$

One calculation

$$ad(\begin{bmatrix} h_1 & 0 & 0 \\ 0 & h_2 & 0 \\ 0 & 0 & h_3 \end{bmatrix})E_{12} = (h_1 - h_2)E_{12}$$

In other words, define

$$\alpha_{12} : \mathfrak{G} \to \mathbb{C}$$

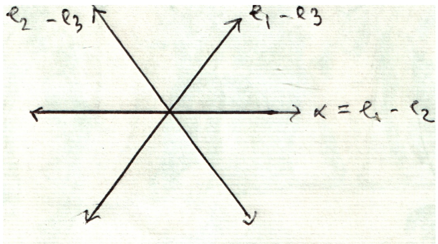
defined by $\alpha_{12}\begin{bmatrix} h_1 & 0 & 0 \\ 0 & h_2 & 0 \\ 0 & 0 & h_3 \end{bmatrix} = h_1 - h_2$. Then, $E_{12} \in \mathfrak{g}^{\alpha_{12}}$.
Set $R = \{\alpha_{ij} | 1 \leq i \neq j \leq 3\}$, then $\mathfrak{g}_{\alpha_{ij}} = \mathbb{C}E_{ij}$. The decomposition

$$\mathfrak{g} = \mathfrak{G} \oplus (\bigoplus_{\alpha \in R} \mathfrak{g}^{\alpha})$$

is simply

$$\mathfrak{g} = \begin{bmatrix} * & 0 & 0 \\ 0 & * & 0 \\ 0 & 0 & * \end{bmatrix} \oplus \left(\bigoplus_{i,j} \mathbb{C}E_{ij} \right)$$

Still missing Euclidean structure on R. We want to draw:



Killing form Given $X, Y \in \mathcal{G}$ define

$$(X|Y) = Trace(ad(X)ad(Y))$$

This defines a symmetric bilinear form on $\mathfrak{G}$. Set

$$\mathfrak{G}_0 = \{X \in \mathfrak{G} : \alpha(X) \in \mathbb{R}, \forall \alpha \in R\}$$

Theorem $(.,.)$ restricts to positive defined inner product on $\mathfrak{G}_0$.

Exercice 9 Compute this explicitly for $\mathfrak{sl}(3, \mathbb{C})$ and $\mathfrak{sl}(n, \mathbb{C})$. Roughly described,

$\mathfrak{g}$- complex simple Lie Algebras	$\rightarrow$	Root System $R \subseteq V \subseteq \mathfrak{G}^*$.

Need to check

1. Differente choices of $\mathfrak{G}$ lead to equivalent root systems.
2. If $\mathfrak{g}_1 \ncong \mathfrak{g}_2 \Rightarrow R_1 \ncong R_2$ (injective)
3. It's surjective.

Theorem *(Serre Relations)* Given a root system R, choose $R^* \subseteq R$ and writte $\pi = \{\alpha_1, ..., \alpha_n\}$ for the simple roots. There is (up to isomorphism) a unique complex simple Lie Algebra $\mathfrak{g}$ with elements $h_i, e_i, f_i, i = 1, ..., n$, and relations

- $[h_i, h_j] = 0$
- $[h_i, e_j] = [\alpha_i, \alpha_j]e_j$
- $[h_i, f_j] = -[\alpha_i, \alpha_j]f_j$
- $[e_i, f_j] = \delta_{ij}h_i$.

$ad(e_i)^{-(\alpha_i,\alpha_j)+1}e_j = 0$, $i \neq j$
$ad(f_i)^{-(\alpha_i,\alpha_j)+1}f_j = 0$, for $i \neq j$. This provides the inverse.

Root System	$\mathfrak{g}$
A_n	$\mathfrak{sl}(n+1, \mathbb{C})$
D_n	$Sl_{2n}(\mathbb{C}) = 2n \times 2n$ skew symmetric matrices
E_6, E_7, E_8	no easy description
F_4	$\mathfrak{so}_{2n+1}(\mathbb{C})$
C_n	$S_p(2n, \mathbb{C})$