Continuous symmetry and Lie algebras I

This week:

① Classify all finite-dimensional $\mathbb{C}$ Lie algebras

A Lie algebra $\cancel{\mathfrak{g}}\, G$ is a complex vector space,
with a bilinear map:

$$\cancel{[\cdot,\cdot]: \mathfrak{g}\times\mathfrak{g}\longrightarrow \mathfrak{g}\ \text{such}}$$

$$[\cdot,\cdot]: G\times G \longrightarrow G \quad \text{such that}$$

$$[X, Y] = -[Y, X]$$

$$[X,[Y,Z]] + [Z,[X,Y]] + [Y,[Z,X]] = 0$$

② Classify "all" representations of G.

A $\mathbb{C}$-representations of G is a linear map

$\Pi: G \longrightarrow M_n(\mathbb{C}) :=$ $n\times n$ complex matrices

such that

$$\Pi([X,Y]) = \Pi(X)\Pi(Y) - \Pi(Y)\Pi(X)$$

Why do we care? In mathematics always en counter: group actions

$$\underset{\text{group}}{G} \times \underset{\text{space}}{X} \longrightarrow X$$

$$(g, x) \longrightarrow g\cdot x$$

$$ex = x$$

$$g\cdot(h\cdot x) = (gh)x$$

Theme 1 of Rep. Theory (RT): Fix a class of objects X, ask what symmetries G can arise.

Ex. (Klein)

$SO(3) = 3\times 3$ orthogonal matrices acts on $\mathbb{R}^3$, preserving length.

Fix $X \subset \mathbb{R}^3$ such that

$$G_X := \{ g \in SO(3) \mid g(X) = X \}$$

is finite.

Ask: what finite groups arise this way? (eg. D_n, $X = \square$)

"Answer: simply laced Dynkin diagrams"

Ex. $GL(n,\mathbb{R})$ acts on $\mathbb{R}^n$

Fix $X \subset \mathbb{R}^n$, such that

$$G_X = \{ g \in GL(n,\mathbb{R}) \mid gX = X \}$$

[$GL(n,F) :=$ $n\times n$ invertible matrices, entries in F]

G_X is $\overbrace{\text{closed, bounded}}^{\text{compact}}$, and connects in subspace topology in $\mathbb{R}^{n^2}$

Answer (range over n)

"The same Dynkin diagrams"

Ask for direct connection: McKay Correspondence

<u>Theme 2</u>: Classify all representations of ~~the~~ a fixed class of G, ie, ~~homomorf~~ homomorphisms:

$$\begin{array}{c} \pi : G \longrightarrow GL(V) \\ \qquad\quad \cong \\ \qquad\quad GL(n, \mathbb{C}) \end{array}$$

Reason this is interesting: Given G acting on X, consider:

$$\mathcal{F}(X) := \{ f : X \longrightarrow \mathbb{C} \}$$

get a repn $\pi : G \longrightarrow GL(\mathcal{F}(X))$

$$g \longmapsto \pi(g)$$

$$(\pi(g)f)(x) = f(g^{-1}x)$$

$\mathcal{F}(X)$ is reducible (eg constant functions are an invariant subspace). Idea: Decompose $\mathcal{F}(X)$ into irreducible repns, learn something about X.

<u>Ex.</u> Quantum mechanics.

To a class real mechanical system X (with symmetry group G_X), get a Hilbert space of states of ~~X~~ $\mathcal{H}$, like functions on X. $\mathcal{H}$ become a representation of G_X. Main equation of QM describes states $|\psi\rangle$ with fixed energy E:

$\cancel{\mathbb{H}|\psi\rangle = E|\psi\rangle}$

(*) $\mathbb{H}|\psi\rangle = E|\psi\rangle$

schroedinger operation

$\pi(g): \mathcal{H} \longrightarrow \mathcal{H}$

$\mathbb{H}: \mathcal{H} \longrightarrow \mathcal{H}$ respects group representation:

$\mathbb{H}\,\pi(g) = \pi(g)\,\mathbb{H} \quad \forall g \in G_x$

So, if $|\psi\rangle$ satisfies (*), then

$$\boxed{\mathbb{H}(\pi(g)|\psi\rangle) \quad E(\pi(g)|\psi\rangle)}$$

$$= \pi(g)\mathbb{H}|\psi\rangle = \pi(g)E|\psi\rangle$$

the E-eigenspace of $\mathbb{H}$ is an invariant subspace of repn of G_x on $\mathcal{H}$

Possible to use the classification of irred rep. of G_x to deduce .... restrictions on E

these lectures are mostly interested in groups

$G \subset GL(n, F)$, $F = \mathbb{R}, \mathbb{C}$

theres a powerful tool.

$\exp: M_n(F) \longrightarrow M_n(F)$

$$(+)\quad \exp(X) := \sum_{k\geq 0} \frac{1}{k!} X^k$$

There's a nice norm on $M_n F$:

entries of X

$$\|X\|^2 = \sum_{1\leq i,j\leq n} |X_{ij}|^2$$

(+) converges in this norm.

Has many of the usual properties of e^x, $x \in \mathbb{R}$:

(1) $\exp(-X)\exp(X) = Id$

(Pf: formally manipulate product.)

(2) $\exp(X)\exp(Y) \neq \exp(X+Y)$

FAILS

(In fact: $\exp(aX)\exp(bY) = \exp(aX+bY)$
$\forall a,b \in F \iff [X,Y] = 0$
$:= XY - YX$)

We're interest in solving

$$\exp(X)\exp(Y) = \exp(Z)$$

So $\quad Z = \log(\exp(X)\exp(Y))$

$$\log(g) := (g - Id) - \frac{1}{2}(g-Id)^2 + \frac{1}{3}(g-Id)^3 + \dots$$

Converges in $\|\cdot\| \iff \|g - Id\| < 1$

Lemma

$(+)$ $X = \log(\exp(X))$ if $\|X\| < \log 2$

Proof: $\|e^X - I\| \leq e^{\|X\|} - 1$

$e^{\|X\|} - 1 \leq 1 \iff \|X\| < \log 2$

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So just formalipate series to prove $(+)$

So conclude: exp is a nice bijection

$$\left\{\begin{matrix}\text{neibourhood of} \\ 0 \text{ in } M_n(F)\end{matrix}\right\} \longrightarrow \left\{\begin{matrix}\text{nbhd of } Id \\ \text{in } Gl(n, F)\end{matrix}\right\}$$

$F = \mathbb{R}, \mathbb{C}$

Try to make $Z = \log(\exp(X) \cdot \exp(Y))$

explicit in this nbhd.

$$Z = \sum_{k \geq 1} \frac{(-1)^{k+1}}{k} \left[\left(1 + X + \frac{X^2}{2!} + \dots\right)\left(1 + Y + \frac{Y^2}{2} + \dots\right) - 1\right]^k$$

$$= \underbrace{X + Y + XY}_{k=1} \underbrace{- \frac{1}{2} XY - \frac{1}{2} YX}_{k=2} + \dots$$

$$= X + Y + \frac{1}{2}[X, Y] + \dots$$

Theorem (Campbell-Baker-Hausdorff)

All terms on RHS can be expressed as ~~iterate~~ iterated brackets of X & Y

(Dykin was first to make this explicit)

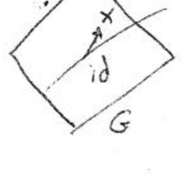
The bracket operation on $M_n(F)$ determines the group multiplications in nbhd of identity in $Gl_n(F)$

Consider: $G \subset GL(n, F)$ subgroup.

Want to define a Lie algebra ~~of~~ ~~some~~ $\mathfrak{g}$ (over F) attached to G

(Eg) $G = GL(n, F)$ $\mathfrak{g} = \mathfrak{gl}(n, F) = M_n(F)$ with $[X,Y] = X$

$$\mathfrak{g} = \{ X \in \mathfrak{gl}(n, F) \mid \exists \text{ a differentiable map } \gamma : \underset{\subseteq \mathbb{R}}{(-\varepsilon, \varepsilon)} \longrightarrow G \text{ s.t. } \left.\frac{d}{dt}\right|_{t=0} \gamma(t) = X \}$$



Lemma

(1) $\alpha X + \beta Y \in \mathfrak{g}$ $\quad \forall \alpha, \beta \in F$

$\forall X, Y \in \mathfrak{g}$

(2) If $X, Y \in \mathfrak{g}$

$[X, Y] := XY - YX \in \mathfrak{g}$

ie, $\mathfrak{g}$ is a lie algebra.

Proof

$\alpha, \beta \in \mathbb{R}$

(1) $X = \left.\frac{\partial}{\partial t}\right|_{t=0} \gamma_X(t)$

$Y = \left.\frac{\partial}{\partial t}\right|_{t=0} \gamma_Y(t)$

Set $\gamma(t) = \gamma_X(\alpha t)\, \gamma_Y(\beta t)$

$\left.\frac{\partial}{\partial t}\right|_{t=0} \gamma(t) = \alpha X + \beta Y \in \mathfrak{g}$

(2) Consider $\gamma(s,t) = \gamma_X(s)\, \gamma_Y(t)\, \gamma_X(s)^{-1}$

$\left.\frac{\partial}{\partial s}\right|_{s=0} \left.\frac{\partial}{\partial t}\right|_{t=0} \gamma = [X, Y]$

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Theorem

There's a ~~bij~~ bijection

$$\left\{\begin{array}{l}\text{connected}\\ \text{subgroups of } GL(n,F)\end{array}\right\} \longleftrightarrow \left\{\begin{array}{l}\text{Lie subalgebras}\\ \text{of } \mathfrak{gl}(n,F)\end{array}\right\}$$

$$G \longmapsto \mathfrak{g}$$

$$\begin{array}{l}\text{group generated}\\ \text{by } \exp(\text{nbhd of } 0 \text{ in } \mathfrak{g})\end{array} \longleftarrow \mathfrak{g}$$