This week

1) Classify all finite dimensional Lie ~~alg~~ algebras.

A Lie algebra ~~of~~ ~~opera~~ $\mathfrak{g}$ is a complex vector space with a bilinear map $[,]: \mathfrak{g}\times\mathfrak{g} \to \mathfrak{g}$ such that $[X,Y]=-[Y,X]$ and
$[X,[Y,Z]]+[Y,[Z,X]]+[Z,[X,Y]]=0$

2) Classify all representations of $\mathfrak{g}$.

A $\mathbb{C}$-representation of $\mathfrak{g}$ is a linear map $\pi: \mathfrak{g} \to M_n(\mathbb{C})$ such that
$\pi([X,Y]) = \pi(X)\pi(Y) - \pi(Y)\pi(X)$

Why do we care? In Mathematics we always encounter group actions:

$$\begin{aligned} G\times X &\longrightarrow X \\ (g,\omega) &\longmapsto g\cdot\omega \end{aligned} \quad (\text{such that } e\cdot\omega=\omega \text{ and } g\cdot(h\cdot\omega)=(g\cdot h)\cdot\omega)$$

Theme 1 of Representation theory: fix a class of objects X and ask which symmetries of G can arise.

<u>Example:</u> $SO(3)$ acts on $\mathbb{R}^3$, preserving length

Fix $X\subset\mathbb{R}^3$ such that $G_X=\{g\in SO(3): g(X)=X\}$ is finite.

~~Wh~~ Which finite groups arise this way?

<u>Example:</u> if X is the square, then $G_X=D_4$.

<u>Answer:</u> ~~S~~ Simply laced Dynkin diagrams.

<u>Example:</u> $GL(n,\mathbb{R})$ acts on $\mathbb{R}^n$. Fix $X\subseteq\mathbb{R}^n$ such that $G_X=\{g\in GL(n,\mathbb{R}): g(X)=$ is ~~a closed set~~ a compact and connected subset of $\mathbb{R}^{n^2}$.

Theme 2: classify all $\mathbb{C}$-representations of a fixed class of G, i.e., homomorphisms $\pi: G \longrightarrow GL(V)\cong GL(n,\mathbb{C})$

Why is this interesting? Given G acting on X, consider $f(X)=\{f:X\to\mathbb{C}\}$.

Get a representation
$$\begin{aligned} \pi: G &\longrightarrow GL(f(X)) \\ g &\longmapsto \pi(g) \end{aligned}$$
$$(\pi(g)f)(\omega)=f(g^{-1}\cdot\omega)$$

~~So~~ $f(X)$ is reducible (for example, the ~~subset of~~ subspace of constant functions is invariant).

Idea: decompose $f(X)$ into irreducible representations.

Example: Quantum mechanics

To a classical mechanical system X (with symmetry group G_X), get a Hilbert space of states of X, $\mathcal{H}$, like functions on X.

$\mathcal{H}$ becomes a representation of G_X

The main equation of quantum mechanics describes the states $|\psi\rangle$ with fixed energy E.

$$\mathbb{H}|\psi\rangle = E|\psi\rangle$$

(Schroedinger operation)

$$\mathbb{H} : \mathcal{H} \longrightarrow \mathcal{H} \text{ respects group representation:}$$

$$\mathbb{H}\pi(g) = \pi(g)\mathbb{H}, \ \forall g \in G_X$$

~~So the E-eigen~~

~~the E-eigenspace of $\mathbb{H}$ is an invariant subspace of~~

these lectures are most interested in groups $G \subset GL(n, F)$, where $F = \mathbb{R}$ or $F = \mathbb{C}$.

There is a powerful tool: $\exp : M_n(F) \longrightarrow M_n(F)$.

$$\exp(X) = \sum_{k=0}^{\infty} \frac{1}{k!} X^k$$

there is a nice norm on $M_n F$:

$$\|X\|^2 = \sum_{i=1}^{n} \sum_{j=1}^{n} |X_{ij}|^2 \quad (\exp(X) \text{ converges in this norm})$$

The map exp has many of the ~~properties~~ usual properties of e^u, with $u \in \mathbb{R}$.

(1) $\exp(-X)\exp(X) = Id$

but $\exp(X)\exp(Y) = \exp(X+Y)$ fails.

In fact, $\exp(aX)\exp(bY) = \exp(aX+bY)$ if and only if $[X,Y]=0$, that is, $XY=YX$.

We are interested in solving $\exp(X)\exp(Y)=\exp(Z)$.

So $Z=\log(\exp(X)\exp(Y))$, where

$$\log(g)=\sum_{k=1}^{\infty}\frac{(-1)^{k-1}}{k}(g-Id)^k$$ (this converges in $\|\cdot\|$, ~~therefore~~ when $\|g-Id\|<1$)

<u>Lemma:</u> $X=\log\exp X$ if $\|X\|<\log 2$.

($\|e^X-Id\|\le e^{\|X\|}-1$ and $e^{\|X\|}-1<1 \Leftrightarrow \|X\|<\log 2$)

Conclude that exp is a nice bijection between a neighborhood of 0 in $M_n(F)$ and a neighborhood of Id in $M_n(F)$.

$$Z=\log(\exp(X)\exp(Y))=\sum_{k=1}^{\infty}\frac{(-1)^{k-1}}{k}\left[\left(1+X+\frac{X^2}{2!}+\dots\right)\left(1+Y+\frac{Y^2}{2!}+\dots\right)-Id\right]^k=$$

$$=X+Y+\frac{1}{2}[X,Y]$$

<u>Theorem:</u> All the terms on RHS can be expressed as iterated brackets of X and Y.

The bracket operation on $M_n(F)$ determines the group multiplication in a neighborhood of Id in $Gl_n(F)$.

Consider $G\subset Gl_n(F)$ as a subgroup of $Gl_n(F)$. We want to define a Lie Algebra $\mathfrak{g}$ (over F) attached to G.

(If $G=Gl_n(F)$, $\mathfrak{g}=M_n(F)$ with $[X,Y]=XY-YX$).

$\mathfrak{g}=\{X\in M_n(F):\exists\ \gamma:]-\varepsilon,\varepsilon[\to G$ differentiable such that $\frac{d\gamma}{dt}\big|_{t=0}=X\}$

<u>Lemma:</u> (1) $\alpha X+\beta Y\in\mathfrak{g}$ whenever $\alpha,\beta\in F$ and $X,Y\in\mathfrak{g}$.

(2) If $X,Y\in\mathfrak{g}$, then $[X,Y]=XY-YX\in\mathfrak{g}$, therefore $\mathfrak{g}$ is a Lie algebra.

<u>Proof:</u> (1) Suppose $X=\frac{d\gamma_X}{dt}\big|_{t=0}$ and $Y=\frac{d\gamma_Y}{dt}\big|_{t=0}$. Set $\gamma(t)=\gamma_X(at)\gamma_Y(bt)$.

$$\frac{d\gamma}{dt}\Big|_{t=0}=\alpha X+\beta Y$$

(2) Consider $\gamma(s,t)=\gamma_X(s)\gamma_Y(t)\gamma_X(s)^{-1}$.

$$\frac{d}{ds}\Big|_{s=0}\frac{d\gamma}{dt}\Big|_{t=0}=[X,Y]$$

Theorem: there is a bijection between connected subgroups of $GL(n,F)$ and Lie subalgebras of $\mathfrak{gl}(n,F)$, ~~where to a Lie subalgebra~~