① LAST TIME

$$\left\{\begin{matrix}\text{Irreducible}\\ \text{Root systems}\end{matrix}\right\} \xleftrightarrow{1-1} \left\{\begin{matrix}\text{Dynkin}\\ \text{diagrams}\end{matrix}\right\}$$

$$\Big\updownarrow 1\text{-}1$$

$$\left\{\begin{matrix}\mathbb{C}\text{- simple}\\ \text{Lie Algebras}\end{matrix}\right\}$$

Tomorrow

Representation of $\mathfrak{g}$

One idea To understand $\mathfrak{g}$ (w/ complicated brackets) might try to "restrict" to subspace $\mathfrak{h} \subseteq \mathfrak{g}$ such that

$$[X,Y]=0 \quad \forall X,Y \in \mathfrak{h}$$

Notice: $\mathfrak{h}$ is a subalgebra, called abelian.

NOTATION given $X \in \mathfrak{g}$ define

$$ad(X) \in End_{\mathbb{C}}(\mathfrak{g}) = \mathcal{M}(n, \mathbb{C})$$

$n = \dim(\mathfrak{g})$

via $ad(X)\,Y = [X,Y]$

So in fact $\quad ad : \mathfrak{g} \longmapsto \mathcal{M}(\mathfrak{g})$

can check $\quad \underset{\uparrow\ \mathfrak{g}\text{-bracket}}{ad[X,Y]} = \underset{\downarrow\ \text{bracket in } \mathcal{M}(\mathfrak{g})}{[ad(X), ad(Y)]}$ (need Jacobi identity)

Jacobi $\quad [X,[Y,Z]] + [Z,[X,Y]] + [Y,[Z,X]] = 0$

might hope that for $X \in$

$\quad$ ad(X) is a diagonalizable endo. of $\mathfrak{g}$

② Example $\mathfrak{g} = \mathcal{H}(2,\mathbb{C})$ Simple?

multiples of Id $= \left\{ \underbrace{\begin{pmatrix} t & 0 \\ 0 & t \end{pmatrix}}_{= Z_t} : t \in \mathbb{C} \right\}$

$[Z_t, X] = 0 \quad \forall X$

non trivial ideal.

Take $\mathfrak{g} := \quad = \left\{ \begin{pmatrix} h & x \\ y & -h \end{pmatrix} : h, x, y \in \mathbb{C} \right\}$

Take $\mathfrak{h} = \mathbb{C}\cdot H$, $H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

Set $X = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, $Y = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$

So, $\mathfrak{g} := \mathfrak{h} \oplus \mathbb{C}X \oplus \mathbb{C}Y$

Look at $\mathrm{ad}(H) : \mathfrak{g} \mapsto \mathfrak{g}$

Compute

$$\mathrm{ad}(H)X = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} - \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

indeed

$$\mathrm{ad}(H)X = 2X$$

Some calculation $\mathrm{ad}(H)Y = -2Y$

③ Define $\alpha \in \mathfrak{h}^* = \mathrm{Hom}_{\mathbb{C}}(\mathfrak{h}, \mathbb{C})$ by $\alpha(H) = 2$

Define $\mathfrak{g}^{\alpha} = \{ A \in \mathfrak{g} \mid \mathrm{ad}(Z)A = \alpha(Z)A \quad \forall Z \in \mathfrak{h} \}$

$= \mathbb{C}X$

$\mathfrak{g}^{\alpha} = \{ A \in \mathfrak{g} \mid \mathrm{ad}(Z)A = -\alpha(Z)A \} = \mathbb{C}Y$

$R := \{ +\alpha, -\alpha \} \subseteq \mathfrak{h}^*$

$\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\beta \in R} \mathfrak{g}^{\beta}$ $\qquad R : \xleftarrow{-\alpha} \underset{A}{\cdot} \xrightarrow{\alpha}$

Small technical wrinkle (?)

If $\mathfrak{h}$ is a maximal abelian subalgebra of $\mathfrak{g}$, $X \in \mathfrak{h}$

$\mathrm{ad}(X) : \mathfrak{g} \mapsto \mathfrak{g}$ need not to be diagonalizable

Bad example Take $\mathfrak{h} = \mathbb{C}\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ in $\mathfrak{sl}(2,\mathbb{C})$

$$\mathrm{ad}\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}^3 = 0$$

Define A Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ is a subalgebra such that

(1) $\forall X, Y \in \mathfrak{h}$, $\exists N$ such that $\mathrm{ad}(X)^N Y = 0$

② If $Z \in \mathfrak{g}$ $\mathrm{ad}(Z)\mathfrak{h} \subseteq \mathfrak{h}$

$\Rightarrow Z \in \mathfrak{h}$ "self normalizing"

④

• For example $\mathfrak{h} = \mathbb{C}\begin{pmatrix}0&1\\0&0\end{pmatrix}$ in $\mathfrak{sl}(2,\mathbb{C})$ violates ② because

$$\left(\mathrm{ad}\begin{pmatrix}1&0\\0&-1\end{pmatrix}\right)\begin{pmatrix}0&1\\0&0\end{pmatrix} = 2\begin{pmatrix}0&1\\0&0\end{pmatrix} \in \mathfrak{h}$$

Theorem (Cartan) A cartan subalgebra $\mathfrak{h}$ is maximally abelian and $\forall H \in \mathfrak{h}$, $\mathrm{ad}(H)$ is a diagonalizable endom. of $\mathfrak{g}$

Let $R \subset \mathfrak{h}^*$ consists of those $\alpha : \mathfrak{h} \longrightarrow \mathbb{C}$ nonzero, s.t. $\mathfrak{g}^\alpha := \{X \in \mathfrak{g} \mid \mathrm{ad}(H)X = \alpha(H)X\} \neq \cdot$

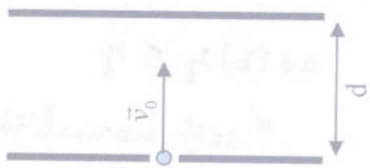
↑ Joint eigenspace of $\{\mathrm{ad}(H) : H \in \mathfrak{h}\}$ with joint eigenvalue α.

Since all of $\mathrm{ad}(H)$ are diagonalizable & commute

$$\mathfrak{g} := \underbrace{\mathfrak{g}^0}_{\mathfrak{h}} \oplus \bigoplus_{\alpha \in R} \mathfrak{g}^\alpha$$

(since $\mathfrak{h}$ is maximally abelian)

Ex: $\mathfrak{sl}(3,\mathbb{C})$ $\quad E_{ij} = \begin{array}{cc} \begin{pmatrix}0&0&0\\0&0&1\\0&0&0\end{pmatrix} & \leftarrow i \\ \uparrow j & \end{array}$



⑤

$$\mathfrak{h} = \left\{ \begin{pmatrix}h_1&0&0\\0&h_2&0\\0&0&h_3\end{pmatrix} : h_1 + h_2 + h_3 = 0 \right\}$$

One calculation

$$\mathrm{ad}\begin{pmatrix}h_1&0&0\\0&h_2&0\\0&0&h_3\end{pmatrix} E_{12} = \begin{pmatrix}h_1&0\\0&h_2\\0&0\end{pmatrix}$$

$$= (h_1 - h_2)E_{12}$$

In other words, define

$$\alpha_{12} : \mathfrak{h} \to \mathbb{C}$$

$$\alpha_{12}\begin{pmatrix}h_1&&\\&h_2&\\&&h_3\end{pmatrix} = h_1 - h_2$$

then $E_{12} \in \mathfrak{g}^{\alpha_{12}}$

Set $R = \{\alpha_{ij} \mid 1 \leq i \neq j \leq 3\}$ then $\mathfrak{g}^{\alpha_{ij}} = \mathbb{C}E_{ij}$

the decomposition $\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha \in R} \mathfrak{g}^\alpha$

is simply

$$\mathfrak{g} = \begin{pmatrix}*&&0\\&*&\\0&&*\end{pmatrix} \oplus \left(\bigoplus_{ij} \mathbb{C}E_{ij}\right)$$

Still missing Euclidian structure on R. We want to draw

$e_2 - e_3 = \beta \longrightarrow$

need more structure

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Killing form   Set $V = \mathbb{R}$-span of $R \subseteq \mathfrak{h}^*$

Given $X, Y \in \mathfrak{h}$ define

$$(X|Y) = \mathrm{Trace}\,(ad(X) \circ ad(Y))$$
$$\in \mathcal{H}(\mathfrak{g}) = \mathfrak{n}_n(\mathfrak{g})$$

this define symmetric bilinear form on $\mathfrak{h}$.

Set $\mathfrak{h}_0 = \{ X \in \mathfrak{h} : \alpha(X) \in \mathbb{R} \quad \forall \alpha \in R \}$

theorem $(\, ,\,)$ restricts to positive definite inner product on $\mathfrak{h}_0$, hence on V^*

Exercise 9 Compute this explicitly for $sl(3,\mathbb{C})$, $sl(n,\mathbb{C})$

(?) Roughly described

$\mathfrak{g}$ - complex simple L. Algebra $\rightsquigarrow$ Root System $R \subseteq V (\subseteq \mathfrak{h}^*)$

Need to check

(1) Different choices of $\mathfrak{h}$ lead to equivalent root systems

(2) If $\mathfrak{g}_1 \not\cong \mathfrak{g}_2 \Rightarrow R_1 \not\cong R_2$ (injective)

(3) It's surjective.

theorem (Serre Relations) Given a root system R, Choose $R^+ \subset R$ and write $\pi = \{\alpha_1, \dots, \alpha_n\}$ for the simple roots.

there is (up to $\cong$) a unique complex simple Lie Algebra $\mathfrak{g}$ with elements h_i, $i = 1, 2, \dots, n$
e_i, $i = 1, 2, \dots, n$
f_i

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and relations
$$[h_i, h_j] = 0$$
$$[h_i, e_j] = (\alpha_i, \alpha_j)\, e_j$$
$$[h_i, f_j] = -(\alpha_i, \alpha_j)\, f_j$$
$$[e_i, f_j] = \delta_{ij}\, h_i$$

$$ad(e_i)^{-(\alpha_i, \alpha_j)+1}\, e_j = 0 \qquad i \neq j$$

$$ad(f_i)^{-(\alpha_i, \alpha_j)+1}\, f_j = 0 \qquad i \neq j$$

this provides the inverse.

ROOT SYSTEM	$\mathfrak{g}$	
A_n	$sl(n+1, \mathbb{C})$	
D_n	$so_{2n}(\mathbb{C}) = 2n \times 2n$	skew-symmetric matrices
E_6, E_7, E_8	no easy description	
G_2, F_4		
	$SO_{2n+1}(\mathbb{C})$	
C_n	$sp(2n, \mathbb{C})$	