Last Trapa lecture.

Representations of $sl(2,\mathbb{C})$ (irreducible).

$$\pi : sl(2,\mathbb{C}) \longrightarrow gl(V) \cong gl(n,\mathbb{C}) \qquad V - \mathbb{C} \text{ vectorspace}$$

such that $\pi([A,B]) = [\pi A, \pi B]$ ← matrix commutator

Examples

① The natural immersion $sl(2,\mathbb{C}) \hookrightarrow gl(2,\mathbb{C})$ is a repn.:

$V = \mathbb{C}^2 \quad v_1 = \begin{pmatrix}1\\0\end{pmatrix} \quad v_{-1} = \begin{pmatrix}0\\-1\end{pmatrix}$

$\pi : sl(2,\mathbb{C}) \hookrightarrow gl(V)$. Let us make this explicit:

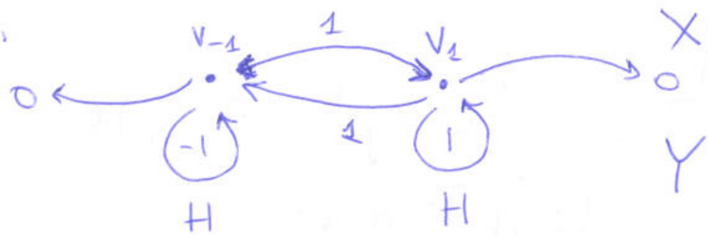
$\pi(H)\cdot v_1 = \begin{pmatrix}1&0\\0&-1\end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix} = \begin{pmatrix}1\\0\end{pmatrix} = 1\cdot v_1.$ (the subscript is the eigenvalue)

$\pi(H)\cdot v_{-1} = -1\cdot v_{-1}.$

$\{v_1, v_{-1}\}$ is a basis of V of eigenvectors for $\pi(H)$.

Moreover, $\pi(X)v_1 = 0 \qquad \pi(X)v_{-1} = v_1$
$\pi(Y)v_1 = v_{-1} \qquad \pi(Y)v_{-1} = 0.$

In pictures:



② The adjoint repn.

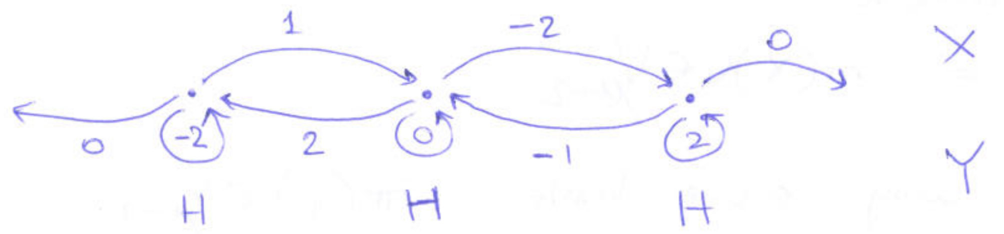
$$\mathrm{ad} : sl(2,\mathbb{C}) \longrightarrow gl(sl(2,\mathbb{C})) \cong gl(3,\mathbb{C})$$

$$A \longmapsto \mathrm{ad}\,A$$

$$\mathrm{ad}\,A(B) = [A,B].$$

$V = \mathfrak{sl}(2,\mathbb{C})$ with basis $\{v_2 = X, v_{-2} = Y, v_0 = H\}$

Picture: v_{-2} v_0 v_2



<u>Remark:</u> A 1-dimensional repn of $\mathfrak{sl}(2,\mathbb{C})$ is isomorphic to the trivial one $V \cong \mathbb{C}$

$\pi(X) = \pi(Y) = \pi(H) = 0.$

(since $\mathbb{C}$ is commutative).

Let us work more generally.

Let V be a fin.-dim. repn of $\mathfrak{sl}(2,\mathbb{C})$

$$\pi : \mathfrak{sl}(2,\mathbb{C}) \longrightarrow \mathfrak{gl}(V).$$

The main idea is that $\pi(H)$ is diagonalisable.

For $\mu \in \mathbb{C}$, set $V_\mu = \{v \in V \mid \pi(H)v = \mu v\}$.

$\pi(H)$ diagonalisable + V fin-dim $\Rightarrow$

$$\Rightarrow V = \bigoplus_{\mu \in \mathbb{C}} V_\mu \quad \underline{\text{finite}} \text{ direct sum.}$$

Let us take $v \in V_\mu$ now, and try to compute $\pi(H)(\pi(X)v)$

(the examples suggest $= (\mu+2)\pi(X)v$).

We can use $\pi(H)\pi(X) - \pi(X)\pi(H) = [\pi(H), \pi(X)] =$

$= [H, X] = 2X$, so that

$$\pi(H)(\pi(X)v) = 2\pi(X)v + \pi(X)\pi(H)v = 2\pi(X)v + \pi(X)\mu v$$

since $v \in V_\mu$, so in the end

$$\pi(H)\big(\pi(X)v\big) = (\mu+2)\,\pi(X)v.$$

We have shown

$$v \in V_\mu \Rightarrow \pi(X)v \in V_{\mu+2}.$$

In the same way, we have $\pi(Y) \in V_{\mu-2}$.

And we can keep going:

$$\cdots V_{\mu-2} \underset{\pi(Y)}{\overset{\pi(X)}{\rightleftarrows}} V_\mu \underset{\pi(Y)}{\overset{\pi(X)}{\rightleftarrows}} V_{\mu+2} \cdots\cdots \rightleftarrows V_\lambda$$

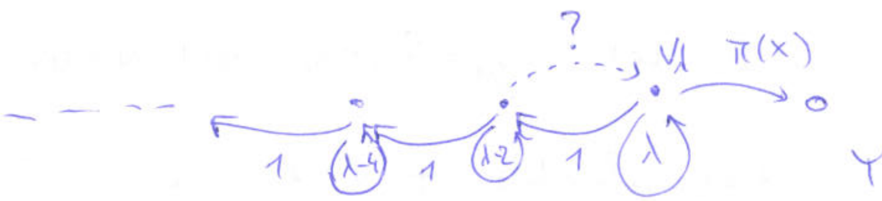
$$v_\lambda \xrightarrow{\pi(X)} 0.$$

the chain stops by finite dimension.

Precisely, fix λ such that $\exists V_\lambda\ \exists v_\lambda \in V_\lambda\ \pi(X)v_\lambda = 0$.

Set $v_{\lambda-2} = \pi(Y)v_\lambda$ and so on.

This chain stops too, but how does X act on $v_{\lambda-2}$?



$\pi(X)v_{\lambda-2} \in V_\lambda$ obviously, but since we have fixed a basis for V, we have to compute the constant with respect to that basis.

$$\pi(X)v_{\lambda-2} = \pi(X)\pi(Y)v_\lambda = \pi(H)v_\lambda - \pi(Y)\underbrace{\pi(X)v_\lambda}_{0} = \lambda v_\lambda.$$

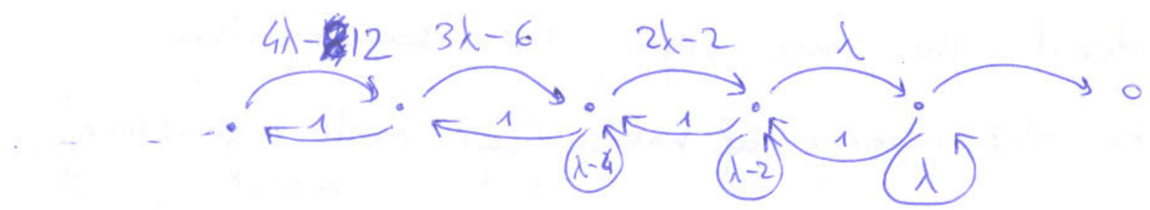
so the dotted arrow becomes

Next we want to compute $\pi(X)v_{\lambda-4}$, and by a similar computation we get

$$\pi(X)v_{\lambda-4} - \pi(Y)\lambda v_\lambda = (\lambda-2)v_{\lambda-2} \quad \text{and so}$$

$$\pi(X)v_{\lambda-4} = (2\lambda-2)v_{\lambda-2}$$

And so on:

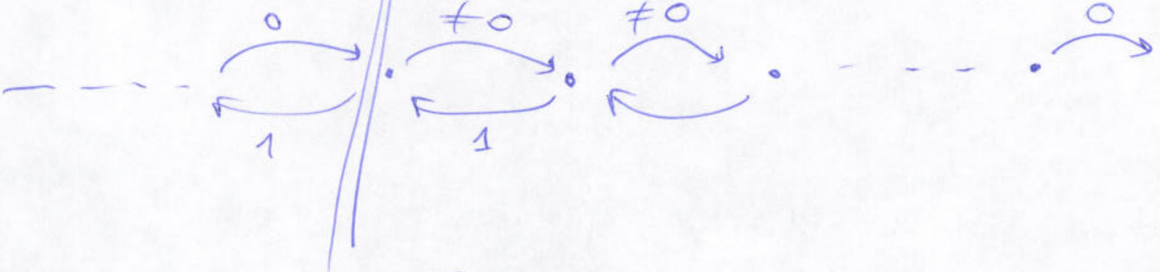


It seems like Y takes the vectors back with no limitations, and that is the case. These formulas define infinite-dimensional repns, called Verma modules (of $sl(2,\mathbb{C})$)

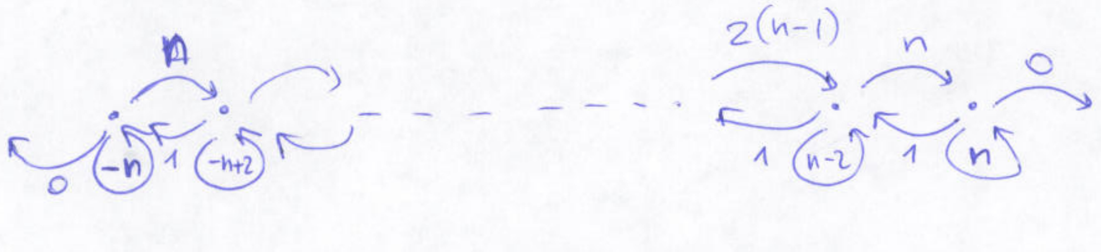
However, we are interested in finite-dimensional repns.

We need $\exists k \in \mathbb{Z}_{\geq 0}\ (k+1)(\lambda-k)=0$.

so that



So the final picture is



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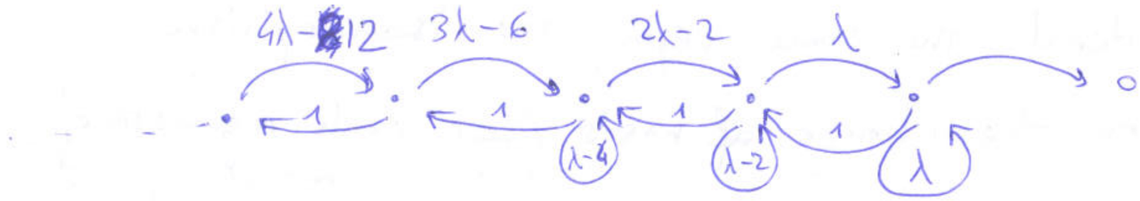

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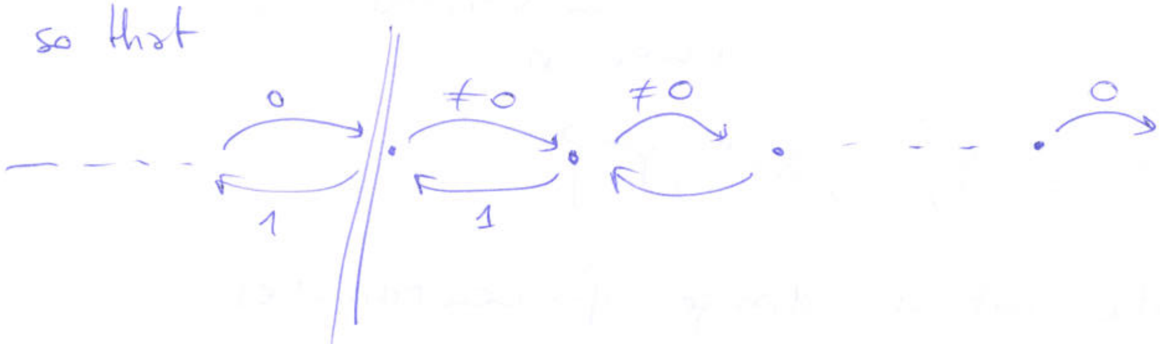

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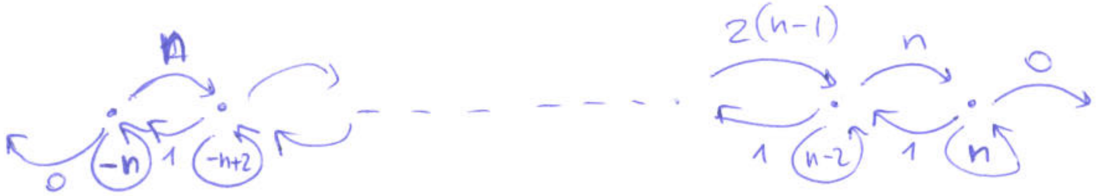
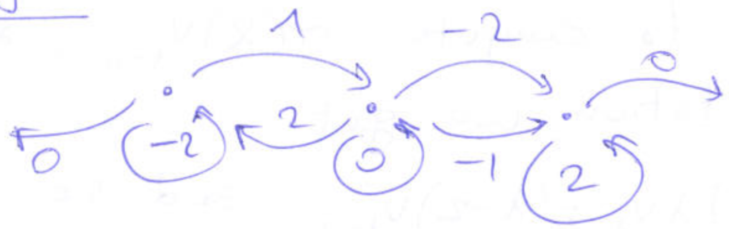
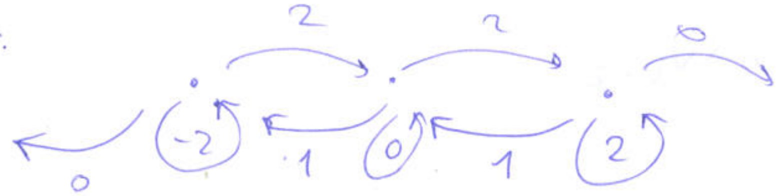


Example, again

Before:



Now:



It is indeed the same repn, the isomorphism is given by the change of basis (for each eigenspace, it is a rescaling).

Theorem

The irreducible repns of $\mathfrak{sl}(2,\mathbb{C})$ are parametrised by $\mathbb{N}$ $\quad n \longmapsto V(n)$

$\dim(V(n)) = n+1$.

Remark

Let $G = SL(2,\mathbb{C})$. $\quad V =$ homogeneous polynomials in 2 variables, of degree n.

$\dim V = n+1$.

$V = \mathrm{span}\{X^n, X^{n-1}Y, \dots, XY^{n-1}, Y^n\}$.

We have the natural change-of-coordinates repn for G. $\quad \tau : G \longrightarrow GL(V)$

We can define

$$d\tau : sl(2,\mathbb{C}) \longrightarrow gl(V)$$

by $d\tau(A)f = \frac{d}{dt}\Big|_{t=0} (\tau(\exp A) f)$.

And that is the $n+1$-dimensional irreducible repn of $sl(2,\mathbb{C})$ (exercise)

The general case

Let $\mathfrak{g}$ $\mathbb{C}$-simple Lie algebra.
Fix $\mathfrak{h}$ Cartan subalgebra $\subseteq \mathfrak{g}$. Define R root system.
choose $R^+ \subseteq R$, and get Π simple roots.
Let $\Lambda^+ \subseteq \mathfrak{h}^*$ $\Lambda^+ := \{\lambda \in \mathfrak{h}^* \mid \frac{2(\lambda,\alpha)}{(\alpha,\alpha)} \in \mathbb{N} \ \forall \alpha \in R^+\}$.

((In $sl(2,\mathbb{C})$, $\mathfrak{h} = \mathrm{span}\{H\}$ $\mathfrak{h}^* \xrightarrow{\sim} \mathbb{C}$ by $\lambda \longmapsto \lambda(H)$))
And $\Lambda^+ \longleftrightarrow \mathbb{Z}_{\geq 0}$.

Theorem (highest weight)

There is a bijection between Λ^+ and irreducible repns of $\mathfrak{g}$ $\lambda \longmapsto V(\lambda)$
and ~~there~~ up to scaling, $\exists!\, v_\lambda \neq 0$ $v_\lambda \in V(\lambda)$
s.t. 1. $Hv_\lambda = \lambda(H)v_\lambda$ $\forall H \in \mathfrak{h}$
2. If $x \in \mathfrak{g}^\alpha$ for $\alpha \in R^+$, then $x \cdot v_\lambda = 0$.