(2) consider $\gamma(s,t) = \gamma_X(s)\,\gamma_Y(t)\,\gamma_X(s)^{-1}$

$$\frac{d}{ds}\Big|_{s=0} \frac{d}{dt}\Big|_{t=0} \gamma = [X,Y]$$

<u>Theorem:</u> There's a bijection:

$$\left\{ \begin{array}{c} \text{connected} \\ \text{subgroups of } GL(n,F) \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{c} \text{Lie subalgebras} \\ \text{of } \mathfrak{gl}(n,F) \end{array} \right\}$$

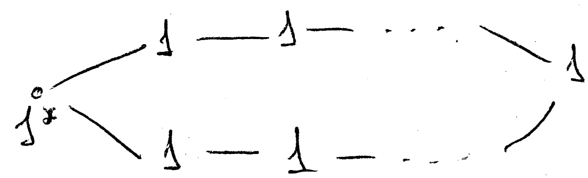
$$G \longmapsto \mathfrak{g}$$

$$\text{group generated by } \exp(\text{nbhd of } 0 \text{ in } \mathfrak{g}) \longleftarrow\!\shortmid\ \mathfrak{g}$$

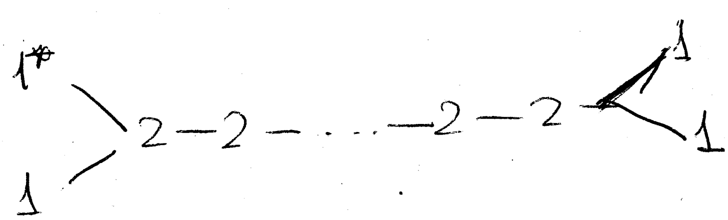
17/07/13

<u>Ejercicio 4:</u>

An



Dn



$$\boxed{\forall \beta: \quad n_\beta = \frac{1}{2} \sum_{\alpha - \beta} n_\alpha}$$

E6

$$\begin{array}{ccccccccc} & & & & 1 & & & & \\ & & & & | & & & & \\ & & & & 2 & & & & \\ & & & & | & & & & \\ 1^* & - & 2 & - & 3 & - & 2 & - & 1 \end{array}$$

E7

$$\begin{array}{ccccccccccccc} & & & & & & 2 & & & & & & \\ & & & & & & | & & & & & & \\ 1^* & - & 2 & - & 3 & - & 4 & - & 3 & - & 2 & - & 1 \end{array}$$

$$\begin{array}{ccccccccccccccc} & & & & & & & & & & 3 & & & & \\ & & & & & & & & & & | & & & & \\ E_8 \quad & 1^* & - & 2 & - & 3 & - & 4 & - & 5 & - & 6 & - & 4 & - 2 \end{array}$$

Today:

no trivial ideals

$\left\{ \begin{matrix} \mathbb{C}\text{-simple} \\ \text{Lie algebra} \end{matrix} \right\} \xleftrightarrow{1-1} \left\{ \begin{matrix} \text{irreducible} \\ \text{root systems} \end{matrix} \right\}$

$\left\{ \begin{matrix} \text{diagram on} \\ \text{Board few} \\ \text{extras} \end{matrix} \right\} \xleftrightarrow{1-1}$

A subalgebra A ($\forall x, y \in A$ $[x,y] \in A$) of a Lie algebra $\mathfrak{g}$ is called an ideal if $\forall x \in \mathfrak{g}, \forall y \in A, [x,y] \in A$

$$[x,y] = -[y,x]$$

A root system is an Euclidean v.s. V (ie, real v.s. with inner product $(,)$) and a finite subset of roots $R \subset V$ st

① R spans V. $0 \notin R$

② If $\alpha \in R$, the only multiples of α in R are $\pm\alpha$

③ Fix α, let s_α be the reflection in hyperplane $\perp$ to α.

Require $s_\alpha(R) = R$ $\forall \alpha \in R$

Formula for s_α:

$$s_\alpha : V \longrightarrow V \quad \text{linear}$$

$$s_\alpha(v) = v - \frac{2(\alpha, v)}{(\alpha,\alpha)} \alpha$$

Check: $s_\alpha(\alpha) = \alpha - 2\alpha = -\alpha$

if $v \perp \alpha$: $s_\alpha(v) = v$

(4) $\frac{2(\alpha,\beta)}{(\alpha,\alpha)} \in \mathbb{Z} \quad \forall \alpha,\beta \in R$

If all α's have same length, say R is simply laced.

The ADE diagrams are called simply laced.

Two root systems $R \subset V$, $R' \subset V'$ are equivalent if $\exists$ a linear $\cong$

$$\Phi: V \to V' \quad \text{st} \quad \Phi(R) = R' \quad \&$$

$$\frac{2(\alpha,\beta)}{(\alpha,\alpha)} = \frac{2\,(\Phi(\alpha),\Phi(\beta))'}{(\Phi(\alpha),\Phi(\alpha))'} \quad \forall \alpha,\beta \in R$$

A root system $R \subset V$ is reducible if $\exists$ root systems $(R_1, V_1), (R_2, V_2)$ st $R = R_1 \amalg R_2$ & $V = V_1 \oplus V_2$

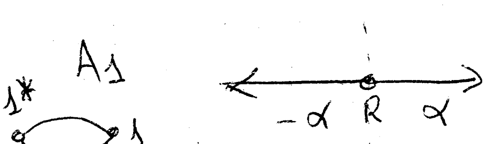
Otherwise, $R \subset V$ is irreducible.

<u>Thm:</u> (1) There's a bijection between the diagrams (A_n, D_n, E_6, E_7, E_8) and irreducible simply laced root systems up to equivalence.

(Will treat nonsimply laced case too).

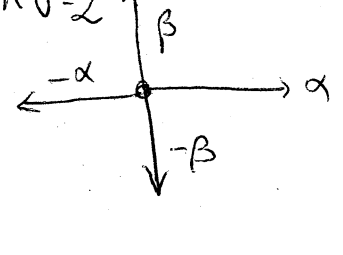
<u>Examples:</u>

1) $\dim(V) = 1$



$\subset \mathbb{R} \cong V$

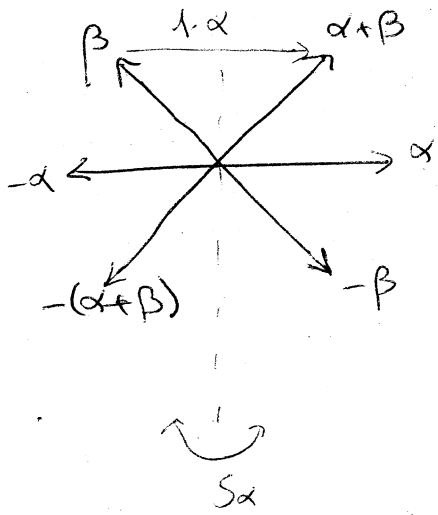
2) $\dim V = 2$



$\subset \mathbb{R}^2$

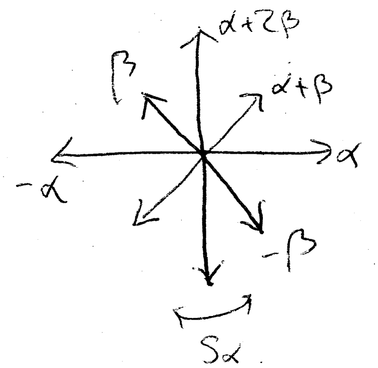
reducible = "$A_1 \oplus A_1$"

3) $\dim V = 2$



A_2

Ex: $\dim V = 2$



not ~~sy~~ simply laced.

To proof th ① need to

harder • Given graph Γ, build $(R_\Gamma \subset V_\Gamma)$

earlier • Given $(R \subset V)$, build Γ

(Closely related to Etingof's exercise 5.3)

Question: if $R \subset V$ is simply laced and $\alpha_1, \dots, \alpha_n \in R$ are a basis for V is it true that $W \cdot \{\alpha_1, \dots, \alpha_n\} = R$

where $W = \langle S_\alpha \mid \alpha \in R \rangle$ weyl group of R

Is W finite?

Knows W acts on R. So get a map $W \longrightarrow$ Permutation of R
$= \{f : R \to R \mid f \text{ biject}$

$(\# R)!$

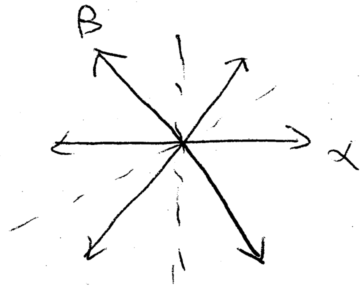
In fact $W \hookrightarrow \mathrm{Per}(R) \Rightarrow \# W < (\# R)!$

If w_1, w_2 have same image then $w_1^{-1} w_2$ acts trivially on R

since R spans V, $w_1^{-1} w_2 =$ Identity linear transf of $V \Rightarrow$

$\Rightarrow w_1 = w_2$

Ex. A_2



What is W?

$s_\beta s_\alpha =$ clockwise rotation by $2\pi/3$

$s_\alpha s_\beta =$ counterclockwise rotation by $2\pi/3$

$W \cong S_3$

<u>Remark:</u> Coxeter presentation of W

R_Γ simply laced $\longleftrightarrow$ diagram Γ

Then $W \cong \langle s_v \mid v$ vertex of Γ, $v \neq 1^*$ $\rangle$

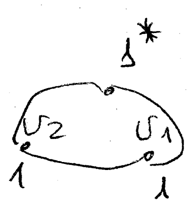
$s_v^2 = id$

$s_{v_1} s_{v_2} = s_{v_2} s_{v_1}$ if v_1 not connected to v_2

$(s_{v_1} s_{v_2})^3 = id$ if $v_1 \to v_2$

<u>Ex:</u>

A_2



$W = \langle s_{v_1}, s_{v_2} \mid s_{v_1}^2 = s_{v_2}^2 = id, (s_{v_1} s_{v_2})^3 = id \rangle$

<u>Ex:</u> $\# W(E8) \simeq 10^9$

Fix diagram $\Gamma \rightsquigarrow$ building $R_\Gamma \subset V_\Gamma$.

Define: $M_\Gamma = \bigoplus_{\substack{\alpha \text{ vertex} \\ \text{of } \Gamma}} \mathbb{Z} f_\alpha$ free $\mathbb{Z}$ module with basis f_α

Define bilinear form

$$(\,,\,): M_\Gamma \times M_\Gamma \longrightarrow \mathbb{Z}$$

$$(f_{v_1}, f_{v_2}) = \begin{cases} 2 & v_1 = v_2 \\ -1 & v_1 \longrightarrow v_2 \quad \text{(connected)} \\ 0 & \text{else} \end{cases}$$

This form has a radical.

Set $f = \sum_{v \text{ vertices}} n_v f_v \in M_\Gamma$

labels

compute

$$(f, f_{v'}) = \left(\sum n_v f_v, f_{v'}\right) = n_{v'} \overset{2}{(f_{v'}, f_{v'})} + \sum_{v - v'} n_v \overset{-1}{(f_v, f_{v'})} =$$

$$= 2n_{v'} - \sum_{v - v'} n_v = 0$$

$\mathrm{rad}(\cdot,\cdot) = \{ e \mid (e,x) = 0 \ \forall x \}$

So span$(f) \subset \mathrm{rad}(\cdot,\cdot)$

In fact span$(f) = \mathrm{rad}(\cdot,\cdot)$

Define $\overline{M}_\Gamma = M_\Gamma / \underset{\text{span}(f)}{\overset{}{\mathrm{rad}(\cdot,\cdot)}}$ inherits nondegerate positive form $(\,,\,)$.

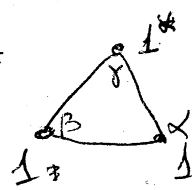
R_Γ = elements of squared length 2 in M_Γ

$$= \{\alpha \in \overline{M_\Gamma} \mid (\alpha,\alpha)=2\} \subset V_\Gamma = \overline{M_\Gamma} \otimes_{\mathbb{Z}} \mathbb{R}$$

We will verify that $R_\Gamma \subset V_\Gamma$ is a simply laced root system.

$$\# W(E8) = 2^{14} 3^5 5^2 7 = 696\,729\,600$$

Ex: $\Gamma =$



$$f = f_\alpha + f_\beta + f_\gamma$$

$$\overline{M_\Gamma} = \{a_\alpha f_\alpha + b_\beta f_\beta + c_\gamma f_\gamma \mid a_\alpha, b_\beta, c_\gamma \in \mathbb{Z}\} \Big/ \{a(f_\alpha + f_\beta + f_\gamma) \mid a \in \mathbb{Z}\}$$

Write $\overline{f_\alpha}$ for image of f_α in $\overline{M_\Gamma}$
similarly for $\overline{f_\beta}, \overline{f_\gamma}$

know $\overline{f_\alpha} + \overline{f_\beta} + \overline{f_\gamma} = 0$

Know $(\overline{f_\alpha}, \overline{f_\alpha}) = (\overline{f_\beta}, \overline{f_\beta}) = (\overline{f_\gamma}, \overline{f_\gamma}) = 2$

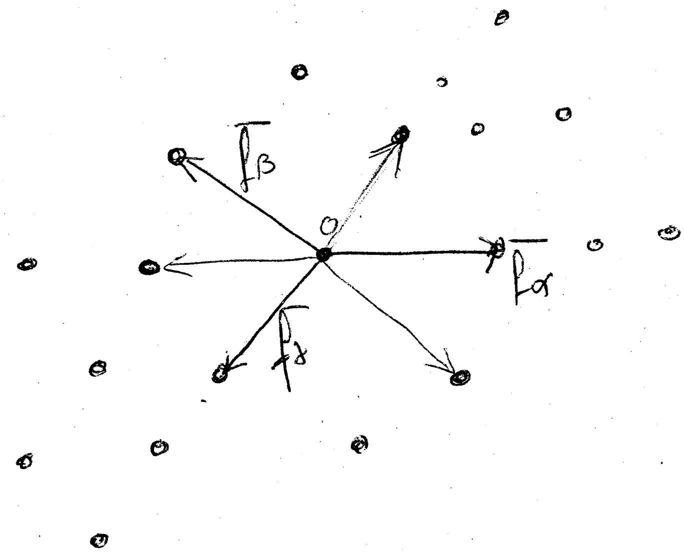
$(\overline{f_\alpha}, \overline{f_\beta}) = (\overline{f_\alpha}, \overline{f_\gamma}) = (\overline{f_\beta}, \overline{f_\gamma}) = -1$

Since: $(x,y) = \|x\| \cdot \|y\| \cos\theta$

Know: the angle between $\overline{f_\alpha}, \overline{f_\beta}, \overline{f_\gamma}$ are all $\frac{2\pi}{3}$

$$\sqrt{2} \cdot \sqrt{2} \cos\left(\frac{2\pi}{3}\right) = -1$$

Picture of $\overline{M_\Gamma}$:



R_Γ = all lattice points of length 2

V_Γ = real span of R_Γ

Rewrite definitions:

$$\overline{M_\Gamma} = \frac{\mathbb{Z}\text{-span of } f_v,\ v \text{ vertex of } \Gamma}{\operatorname{span}(\Sigma n_v f_v)}$$

$\cup$

R_Γ = points of length 2

$$(f_{v_1}, f_{v_2}) = \begin{cases} 2 & v_1 = v_2 \\ -1 & v_1 - v_2 \\ 0 & \text{else} \end{cases}$$

Axioms:

1) R spans V (obvious)

2) only multiples of α are $\pm\alpha$ (obvious)

3) $S_\alpha R = R \ \forall \alpha$

$$S_\alpha(\beta) = \beta - \frac{2(\alpha,\beta)}{(\alpha,\alpha)}\beta = \beta - \underset{\mathbb{Z}}{\underset{\cap}{(\alpha,\beta)}}\beta$$

4) $\frac{2(\alpha,\beta)}{(\alpha,\alpha)} \in \mathbb{Z}$ ✓

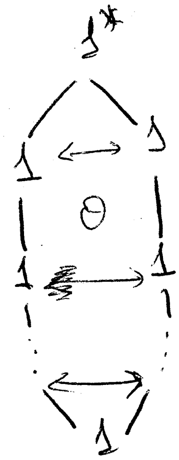
So we've succeeded in building a simply laced root system $(R_\Gamma \subset V_\Gamma)$ from Γ.

Still need to go backwards.

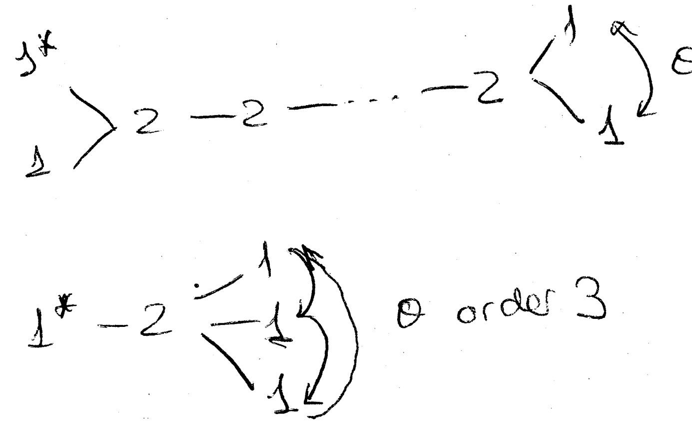
First: nonsimply laced case.

Suppose Γ is one of our diagrams and θ is a graph automorphism of Γ fixing 1^*.

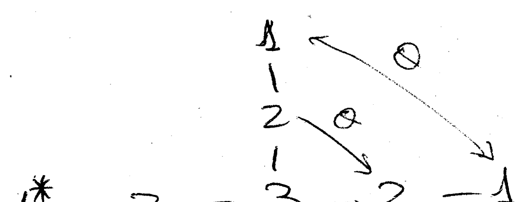
A_n



D_n

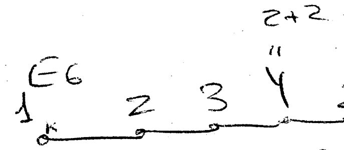


E_6



Build a new graph:

- vertices are orbits of θ
- labels are sums of labels in orbit.



extra label: k_v = size of orbit for vertex v.

Terminology: if $k_v = 1$, say v is a long vertex

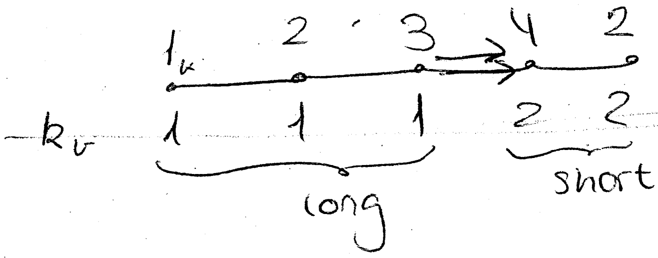
if $k_v > 1$, say v is a short vertex

New edges: if v_1 — v_2 (v_1 short, v_2 long), replace edge by k_α edges pointing to short vertex.

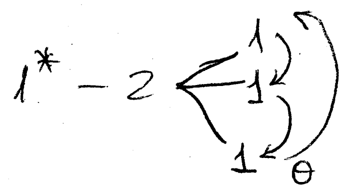
E_6

F4

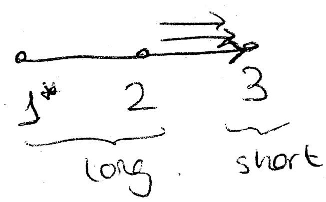
non simply laced affine Dynkin diagram

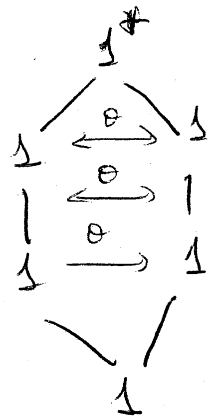


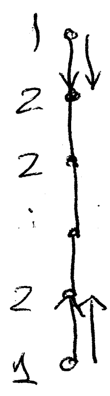
redundant



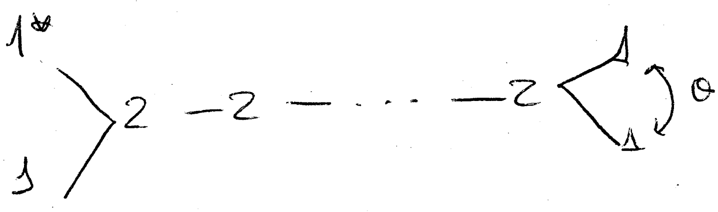
G2

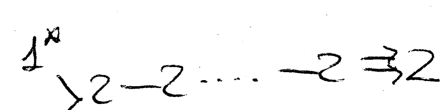






C_n





B_n

In nonsimply laced case, define $\overline{M_\Gamma} = M_\Gamma/\mathrm{rad}(,)$ as befor

$(f_v, f_v) = 2/k_\alpha$

$v \neq v'$ $(f_v, f_{v'}) = -1$ if $v \overset{\text{any \# edges}}{—} v'$

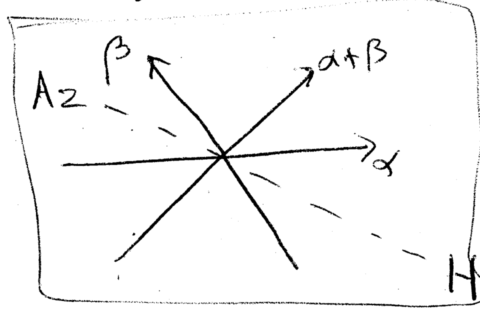
$(f_v, f_{v'}) = 0$ else

Same construction but take R_Γ = lattice points of squared lengt 2 and $\frac{2}{k_\alpha}$

(Throw away higher multiples of roots).

still have to build Γ from a root system. (ie, the inverse construction).

Given a root system $R \subset V$, choose hyperplane $H \subset V$ which doesn't contain any roots.



(Ex)

Get decomposition $R = R^+ \amalg R^-$

$$\boxed{\begin{aligned} R^+ &= \{\alpha, \beta, \alpha+\beta\} \\ R^- &= \{-\alpha, -\beta, -(\alpha+\beta)\} \end{aligned}}$$ (Ex)

Say a root $\gamma \in R^+$ is <u>divisible</u> if $\gamma = t_1 + t_2$ for $t_1, t_2 \in R^+$.

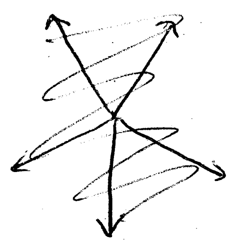
set $\Pi = \{\alpha \in R^+ \mid \alpha \text{ not divisible}\}$

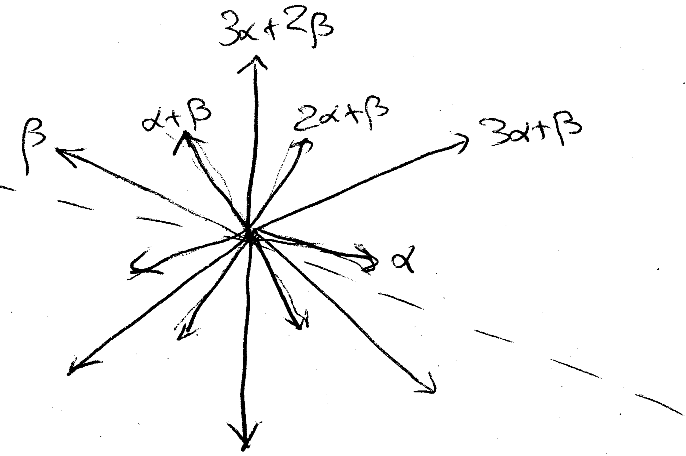
$$\boxed{\Pi = \{\alpha, \beta\}}$$ (Ex)

Th: • Π is a basis for V

• Moreover, any $\gamma \in R^+$ is an integer linear combination of elements of Π.

Ex: G2





$\Pi = \{\alpha, \beta\}$

If $\gamma \in R^+$, write $\gamma = \sum_{\alpha \in \Pi} n_\alpha \alpha$

Define $\boxed{ht(\gamma) = \sum_{\alpha \in \Pi} n_\alpha \in \mathbb{N}}$

Eg. $\alpha \in \Pi$ $ht(\alpha) = 1$

Th: There exists a unique highest root θ in R^+.

write $\theta = \sum_{\alpha \in \Pi} n_\alpha \alpha$

Build a graph: Vertices: $\Pi \cup \{-\theta\}$

Labels on vertices are the coeffs in the expression $0 = 1 (-\theta) + \sum_{\alpha \in \Pi} n_\alpha \alpha$

Edges: connect two vertices α, β if $(\alpha, \beta) \neq 0$

(Determines graph in simply laced case)

Non simply laced: # edges $= -(\alpha,\beta) \cdot \frac{\|\alpha\|^2}{\|\beta\|^2}$ if α is long, β is short

<u>Th:</u> This procedure is well-defined. More precisely W acts transitively on all possible choices of Π.