$\boxed{2}$ Today:

$\left\{ \begin{array}{l} \mathbb{C}\text{-simple} \\ \text{Lie algebras} \end{array} \right\}$ no nontrivial ideals

1-1

$\left\{ \begin{array}{l} \text{Irreducible} \\ \text{Root systems} \end{array} \right\}$

$\left\{ \begin{array}{l} \text{diagram on board} \\ \text{+ few extras (Ex. 4)} \end{array} \right\}$ 1-1

A subalgebra $\mathfrak{a}$ of a Lie algebra $\mathfrak{g}$ is called an ideal if:

$$\forall X \in \mathfrak{g},\ Y \in \mathfrak{a} \quad \underset{-[Y,X]}{\underset{\shortparallel}{[X,Y]}} \in \mathfrak{a}$$

(subalgebra: $\forall X, Y \in \mathfrak{a}$ $[X,Y] \in \mathfrak{a}$)

A root system is a Euclidean vector space V (i.e. real vector space with inner product $(,)$) and a finite subset of roots $R \subset V$ such that:

① R spans V

② If $\alpha \in R$, the only multiples of α in R are $\pm\alpha$

③ Fix α, let S_α be the reflection in hyperplane $\perp$ to α. Require

$$S_\alpha(R) = R \quad \forall \alpha \in R$$

Formula for S_α: $S_\alpha : V \to V$ linear

$$S_\alpha(v) = v - \frac{2(\alpha, v)}{(\alpha,\alpha)}\alpha$$

Check: $S_\alpha(\alpha) = \alpha - 2\alpha = -\alpha$

if $v \perp \alpha$ $S_\alpha(v) = v$

④ $\frac{2(\alpha,\beta)}{(\alpha,\alpha)} \in \mathbb{Z} \quad \forall \alpha, \beta \in R$

If all α's have same length, say R is simply laced.

The ADE diagrams are called simply laced.

Two root system $R \subset V$, $R' \subset V'$ are equivalent if $\exists$ a linear $\cong$ $\phi: V \to V'$ such that

$\phi(R) = R'$ and $\frac{2(\alpha,\beta)}{(\alpha,\alpha)} = \frac{2(\phi(\alpha), \phi(\beta))}{(\phi(\alpha), \phi(\alpha))} \quad \forall \alpha, \beta \in R$

A root system $R \subset V$ is reducible if $\exists$ root systems (R_1, V_1), (R_2, V_2) such that

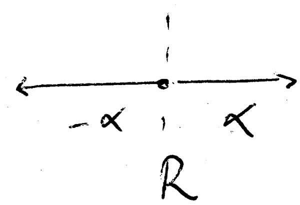
$R = R_1 \amalg R_2$ and $V = V_1 \oplus V_2$

Otherwise $R \subset V$ is irreducible

Theorem: There is a bijection between the diagrams ($A_n, D_n, E6, E7, E8$) and irreducible simply laced root systems up to equivalence.
[Will treat nonsimply laced case too]

Examples:

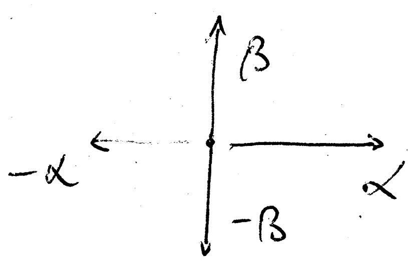
① $\dim V = 1$



$\subset \mathbb{R} \cong V$

A_1

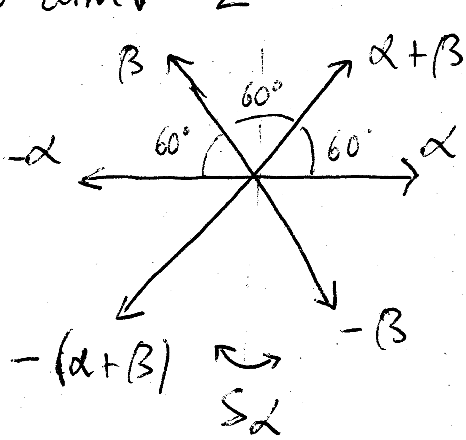
② $\dim V = 2$



$\subset \mathbb{R}^2$

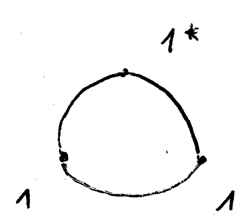
$A_1 \oplus A_1$
$\parallel$
reducible

③ $\dim V = 2$

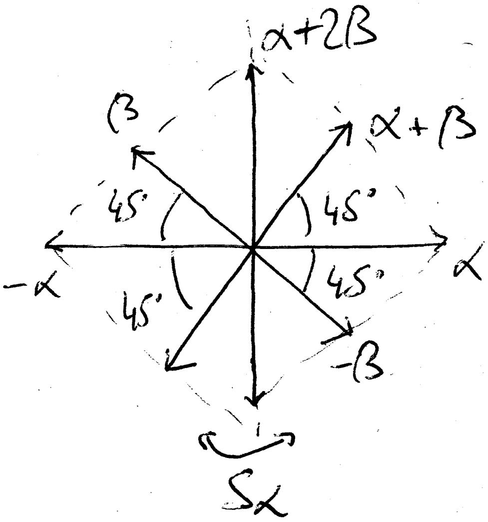


$$S_\alpha(\gamma) = \gamma - \frac{2(\alpha,\gamma)}{(\alpha,\alpha)}\alpha$$

A_2



Ex: $\dim V = 2$



not simply laced

$(\alpha,\alpha) = 2(\beta,\beta)$

To prove this theorem need to

harder ① Given graph Γ, build $(R_\Gamma \subset V_\Gamma)$

easier ② Given $(R \subset V)$, build Γ

(Closely related to Etingof's Exercise 5.3)

Question: If $R \subset V$ is simple laced and $\alpha_1, \alpha_2, \dots, \alpha_n \in R$ are a basis for V, is it true that $W \cdot \{\alpha_1, \dots, \alpha_n\} = R$ where

$W = \langle s_\alpha \mid \alpha \in R \rangle$ Weyl group of R

Q. Is W finite?

Know W acts on R. So get a map

$W \longrightarrow$ Permutation of $R = \{ f: R \longrightarrow R \mid f \text{ bijection} \}$

order $(\#R)!$

In fact $W \hookrightarrow \mathrm{Perm}(R)$, so $\#W < (\#R)!$
If w_1, w_2 have same image, then
$w_1^{-1} w_2$ acts trivially on R.
Since R spans V, $w_1^{-1} w_2$ = Identity linear transformation of V. $\to$ $w_1 = w_2$

Original Q: simply laced

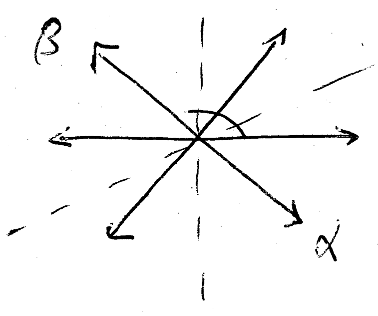
$$W \cdot \{\alpha_1, \ldots, \alpha_n\} \overset{?}{=} R$$

Think (check!) (simply laced)

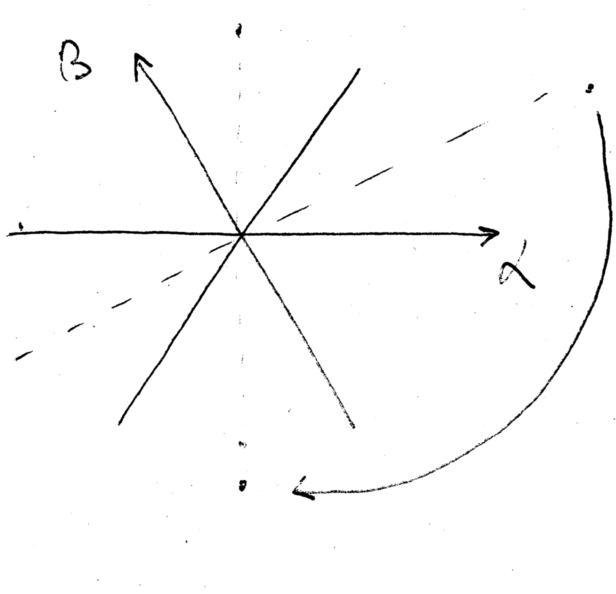
$W \cdot \alpha_1$ is itself a root system $\to$

$\to R = W \cdot \alpha_1 \amalg R_2 \to R$ reduces

<u>Ex</u>: A_2



What is W?



$S_\beta S_\alpha$ = clockwise rotation by $2\pi/3$

$S_\alpha S_\beta$ = counterclockwise rotation by $2\pi/3$

There are order 3 elements, but no order 6 elements $\longrightarrow$ W is $S_3 \cong D_3$

$\# D_3 = 6$.

<u>Remark</u>: Coxeter presentation of W

R_Γ simply laced $\longleftrightarrow$ diagram Γ

<u>Theorem</u>

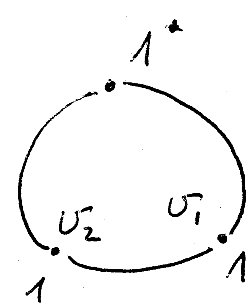
$W \cong \langle S_v,\ v \text{ vertex of } \Gamma,\ v \neq 1^* \rangle$

$S_v^2 = id$, $S_{v_1} S_{v_2} = S_{v_2} S_{v_1}$ if v_1 not adjacent to v_2.

$(S_{v_1} S_{v_2})^3 = id$ if $v_1 - v_2$

<u>Ex.</u>

A_2



$W = \{ S_{v_1}, S_{v_2} \,/\, S_{v_1}^2 = S_{v_2}^2 = id,\ (S_{v_1} S_{v_2})^3 = id \}$

<u>Ex</u>: $\# W(E8) \approx 1 * 10^9$

Fix diagram $\Gamma \rightsquigarrow$ building $R_\Gamma \subset V_\Gamma$

Define $\mathcal{M}_\Gamma = \bigoplus\limits_{\substack{\alpha \text{ vertex} \\ \text{of } \Gamma}} \mathbb{Z} f_\alpha$ free $\mathbb{Z}$ module, basis f_α's

Define bilinear form $(\ ,\ ) : \mathcal{M}_\Gamma \times \mathcal{M}_\Gamma \longrightarrow \mathbb{Z}$

$$(f_{v_1}, f_{v_2}) = \begin{cases} 2 & v_1 = v_2 \\ -1 & v_1 - v_2 \ (v_1 \text{ adjacent to } v_2) \\ 0 & \text{else} \end{cases}$$

This form has a radical

$$\begin{tikzcd}[column sep=small, row sep=small]
\underset{1}{*} \arrow[r, dash] & 2 \arrow[r, dash] & 3 \arrow[r, dash] & 4 \arrow[r, dash] & 5 \arrow[r, dash] & 6 \arrow[r, dash] \arrow[d, dash] & 4 \arrow[r, dash] & 2 \\
 & & & & & 3 & &
\end{tikzcd}$$

Set $f = \sum\limits_{\substack{v \\ \text{vertices}}} \underbrace{n_v}_{\text{labels}} f_v \in \mathcal{M}_\Gamma$

compute

$(f, f_{v_1}) = (\sum n_v f_v, f_{v'}) =$

$= n_{v'} (f_{v'}, f_{v'}) + \sum\limits_{v - v'} n_v n_{v'} (f_v, f_{v'}) =$

$= 2 n_{v'} - \sum\limits_{v - v'} n_v = 0$

$\mathrm{rad}(\cdot,\cdot) = \{ e \,/\, (e,x) = 0 \ \forall x \}$

So $\mathrm{span}(f) \subset \mathrm{rad}(\cdot,\cdot)$

In fact $=$

Define $\overline{M}_\Gamma = M_\Gamma / \mathrm{rad}(\cdot,\cdot) = \mathrm{span}(f)$

Inherits nondegenerate <u>positive</u> form $(\,,\,)$

$(f_v, f_v) = 2$

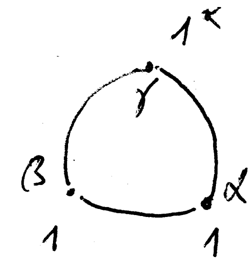
R_Γ = elements of squared length 2 in $\overline{M}_\Gamma$

$= \{ \alpha \in \overline{M}_\Gamma \,/\, (\alpha,\alpha) = 2 \} \subset V_\Gamma = \overline{M}_\Gamma \otimes_{\mathbb{Z}} \mathbb{R}$

After break: verify that $R_\Gamma \subset V_\Gamma$ is a simply laced root system.

$\# W(E8) = 2^{14} 3^5 5^2 7 = 696\ 729\ 600$

<u>Ex</u>

$\Gamma =$ 

$M_\Gamma = \{ a_\alpha f_\alpha + b_\beta f_\beta + c_\gamma f_\gamma \,/\, a_\alpha, b_\beta, c_\gamma \in \mathbb{Z} \}$

$\overline{M}_\Gamma = M_\Gamma / \{ a(f_\alpha + f_\beta + f_\gamma) \,/\, a \in \mathbb{Z} \}$

Write $\overline{f_\alpha}$ for image of f_α in $\overline{M_\Gamma}$

similarly for $\overline{f_\beta}, \overline{f_\gamma}$

know $\overline{f_\alpha} + \overline{f_\beta} + \overline{f_\gamma} = 0$

Know $(\overline{f_\alpha}, \overline{f_\alpha}) = (\overline{f_\beta}, \overline{f_\beta}) = (\overline{f_\gamma}, \overline{f_\gamma}) = 2$

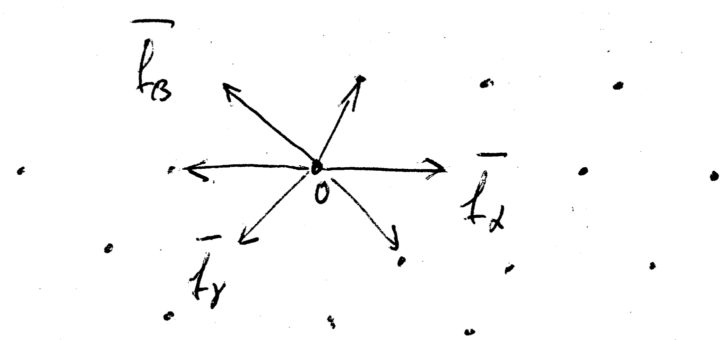
$(\overline{f_\alpha}, \overline{f_\beta}) = (\overline{f_\alpha}, \overline{f_\gamma}) = (\overline{f_\beta}, \overline{f_\gamma}) = -1$

Since $(x, y) = |x||y|\cos\theta$

Know the angle between $\overline{f_\alpha}, \overline{f_\beta}, \overline{f_\gamma}$

are all $2\pi/3$ $\quad \sqrt{2}\sqrt{2}\cos(2\pi/3) = -1$

Picture of $\overline{M_\Gamma}$ lattice of points



R_Γ = all latice point of length 2

V_Γ = real span of R_Γ

Rewrite definitions:

$$\overline{M_\Gamma} = \frac{\mathbb{Z}\text{-span of } f_v,\ v \text{ vertex of } T}{\mathrm{span}(\sum n_v f_v)}$$

$\cup$

R_Γ = points of length 2

$$(f_{v_1}, f_{v_2}) = \begin{cases} 2 & v_1 = v_2 \\ -1 & v_1 - v_2 \\ 0 & \text{else} \end{cases}$$

Axioms

① R spans V

② only multiples of α are $\pm\alpha$

③ $S_\alpha R = R \ \forall \alpha$

$$S_\alpha(\beta) = \beta - \frac{2(\alpha,\beta)}{(\alpha,\alpha)}\alpha = \beta - \underbrace{(\alpha,\beta)}_{\in \mathbb{Z}}\alpha$$

④ $\frac{2(\alpha,\beta)}{(\alpha,\alpha)} \in \mathbb{Z}$

So we've succeeded in building a simply laced root system $(R_T \subset V_T)$ from T.

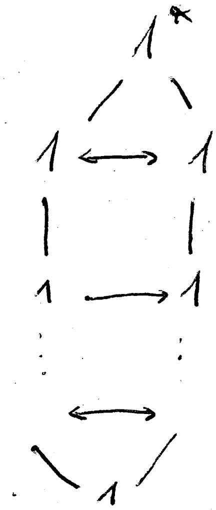
(Notice, in ③ S_α preserves length and we've seen that it also preserves the lattice so $S_\alpha R = R \ \forall \alpha$.)

Still need to go backwards

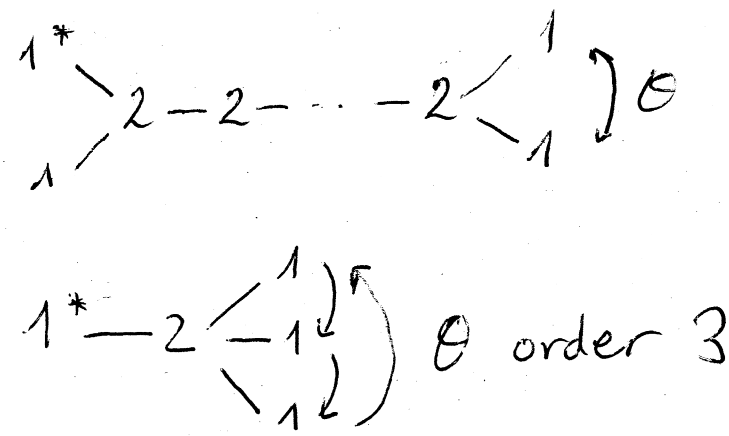
First: nonsimply laced case

Suppose T is one of our diagrams and θ is a graph automorphism of T fixing 1^*.

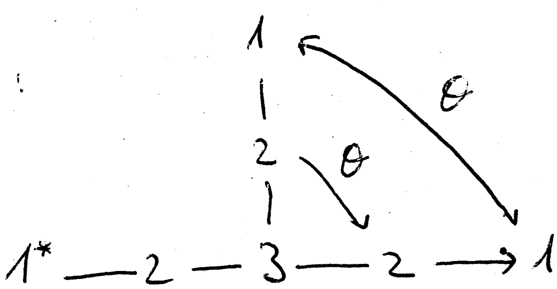
A_n



D_n



E6:



Build a new graph:

- vertices are orbits of θ
- labels are sums of labels in orbit

$$1^* — 2 — 3 — \overset{2+2}{4''} — \overset{1+1}{2''}$$

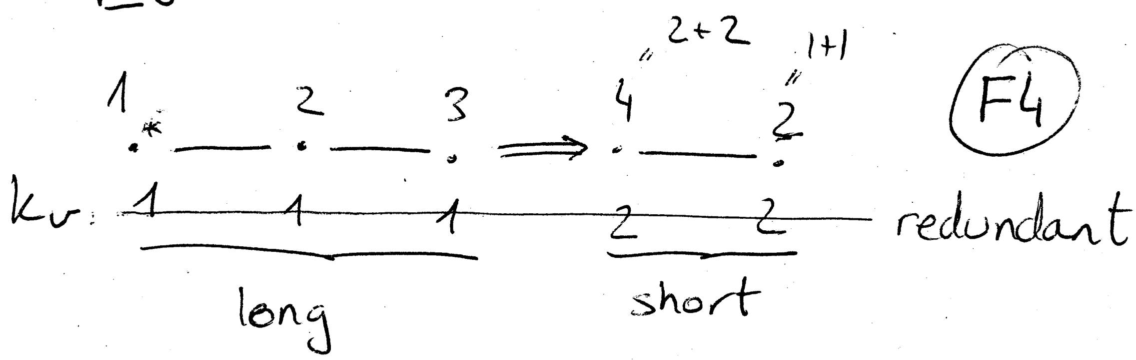
extra label • k_v = size of orbit for vertex v

Terminology: if $k_v = 1$, say v is a long vertex
$k_v > 1$, say v is a short vertex

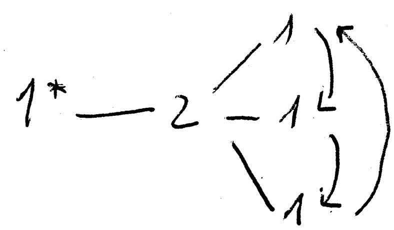
New edges: if $\underset{\text{short}}{v_1} — \underset{\text{long}}{v_2}$, replace edge by

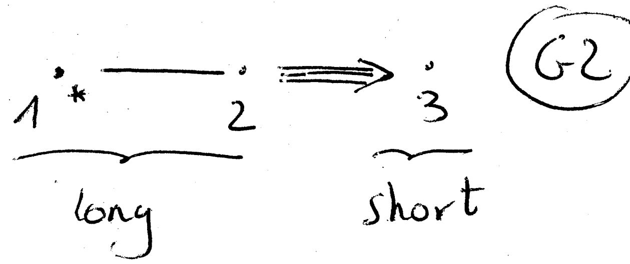
k_α edges pointing to short vertex.

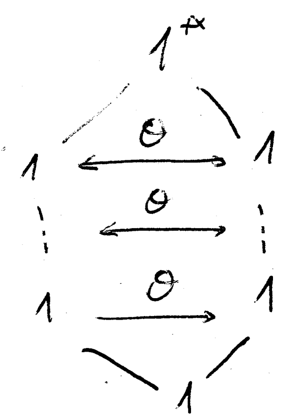
E6

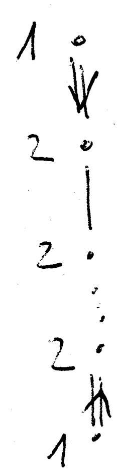


non-simply laced affine Dynkin diagram

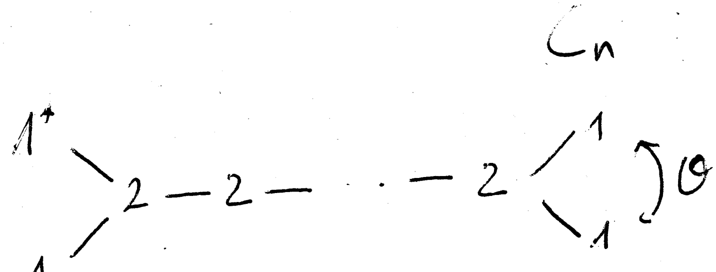


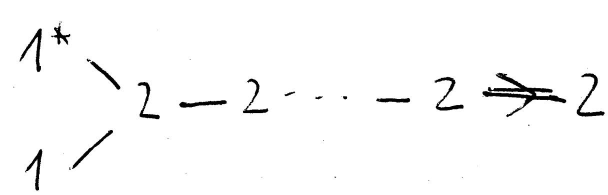






C_n





B_n

In nonsimply laced case, define

$\overline{M_T} = M_T / \mathrm{rad}(\,,\,)$ as before

$(f_v, f_v) = 2/k_v$

any # edges

$v \neq v'$ $(f_v, f_{v'}) = -1$ if v — v'

$(f_v, f_{v'}) = 0$ else

Same construction, but take

R_T = lattice points of squared length

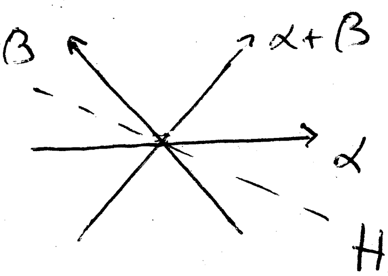
2 & $2/k_\alpha$

(Throw away higher multiples of roots)

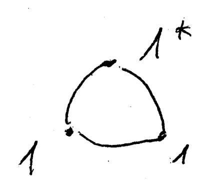
Still have to build T from a root system

(i.e. the inverse construction)

A_2



$R^+ = \{\alpha, \beta, \alpha+\beta\}$



$R^- = \{-\alpha, -\beta, -(\alpha+\beta)\}$

Given $R \subset V$ root system, choose hyperplane $H \subset V$ which doesn't contain any roots

Get decomposition $R = R^+ \amalg R^-$

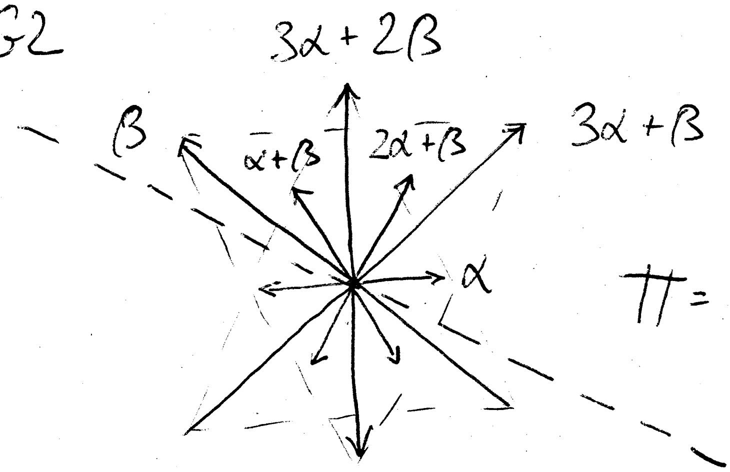
Say a root $\gamma \in R^+$ is divisible if $\gamma = \tau_1 + \tau_2$ for $\tau_1, \tau_2 \in R^+$

Set $\Pi = \{\alpha \in R^+ / \alpha \text{ not divisible}\}$

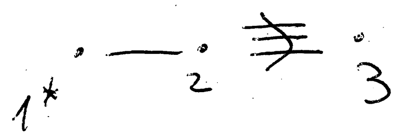
<u>Theorem</u>: • Π is a basis for V

• Moreover any $\gamma \in R^+$ is an integer linear combination of elements of Π

<u>Ex</u> G2



$\Pi = \{\alpha, \beta\}$



If $\gamma \in R^+$, write $\gamma = \sum_{\alpha \in \Pi} n_\alpha \alpha$

Define $ht(\gamma) = \sum_{\alpha \in \Pi} n_\alpha \in \mathbb{N}$

<u>ex</u>. $\alpha \in \Pi$ $ht(\alpha) = 1$

<u>Theorem</u>: There exists a unique highest root θ in R^+. Write $\theta = \sum_{\alpha \in \Pi} n_\alpha \alpha$

Build a graph

vertices $\Pi \cup \{-\theta\}$

labels on vertices are the coeffs in expression $0 = 1 \cdot (-\theta) + \sum_{\alpha \in \Pi} n_\alpha \alpha$.

Edges: Connect two vertices α, β if $(\alpha, \beta) \neq 0$

(Determines graph in simply laced case)

Non simply laced:

$\#\text{edges} = -(\alpha, \beta) \cdot \frac{\|\alpha\|^2}{\|\beta\|^2}$ if α is long, β is short

Theorem: This procedure is well-defined

More precisely W acts transitively on all possible choices of Π