Plane algebraic curves - Problem set for Lecture 1

1. **(Classification of affine conics.)** Show that any conic in $\mathbb{C}^2$ is affinely equivalent to one of the following:

    (a) $X^2 + Y^2 - 1$

    (b) $X^2 - Y$

    (c) $X^2 - Y^2$

    (d) $X^2 - 1$

    (e) X^2

2. Show that the multiplicity of intersection of a plane curve with a line ℓ defined in class does not depend on the choice of linear parametrization for ℓ.

3. Show that the multiplicity of intersection of a plane curve with a line is invariant by affine equivalence.

4. (a) Let ℓ be a line through the origin in $\mathbb{C}^2$. Compute the number of intersection points (counted with multiplicity) of the following plane curves with ℓ. This number will vary with the choice of ℓ. When this number is less than the degree of the curve, try to guess the reason why this happens.

    i. $aX + bY + c$, with $a, b \in \mathbb{R}$, $(a, b) \neq (0, 0)$

    ii. $XY - 1$

    iii. $Y^2 - X^3 - X^2$

    (b) Formulate and prove a criterion for the number of intersection points (counted with multiplicity) of a plane curve with a line to be less than the degree of the curve.