Lecture 1 Plane algebraic curves. Carolina Araujo.

Objects we are interested in.

$f(x,y) \in \mathbb{C}[x,y]$ - UFD.

$\mathcal{Z}(f) = \{p \in \mathbb{C}^2 / f(p) = 0\} \subset \mathbb{C}^2 \}$.

$f \neq const$, $f(x,y) = f_1(x,y)^{m_1} \ldots f_n(x,y)^{m_k}$, $m_i > 0$.

$f_i$ - irreducible.

$f_i, f_j$ - coprime for $i \neq j$.

decomposition is unique.

$\mathcal{Z}(f) = \mathcal{Z}(f_1) \cup \ldots \cup \mathcal{Z}(f_k) \subseteq \mathbb{C}^2$.

$\quad \shortparallel$

$\mathcal{Z}(f_1, \ldots f_k)$.

Example: $f_t(x,y) = ty - x^2$, $t \in \mathbb{C}$.

$C_t = \mathcal{Z}(f_t)$.


$C_0 = \mathcal{Z}(f_0) = \mathcal{Z}(x^2) = \mathcal{Z}(x)$.

Def: An (affine) plane curve is an equivalence ~~relation~~ class of nonconstant polynomials, where $f \sim g$ if $\exists \lambda \in \mathbb{C} \setminus \{0\} : f = \lambda g$.

Let: Curve $f$, $p \in f \longleftrightarrow p \in \mathcal{Z}(f)$.

$f \cap g \longleftrightarrow \mathcal{Z}(f) \cap \mathcal{Z}(g)$.

Examples:

degree 1: $l = ax + by + c$, $a, b \neq 0$.

Def: An affine equivalence of $\mathbb{C}^2$ is a map $\varphi : \mathbb{C}^2 \to \mathbb{C}^2$ of the following type: $\varphi(x,y) = (\varphi_1(x,y), \varphi_2(x,y))$, where

$$\begin{cases} \varphi_1 = a_{11}x + a_{12}y + a_1 \\ \varphi_2 = a_{21}x + a_{22}y + a_2. \end{cases} \quad \det\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \neq 0.$$

Any 2 lines are affine equivalent.

$\varphi : \mathbb{C}^2 \to \mathbb{C}^2$.

$C = \mathcal{Z}(f)$.

$\varphi(C) = \mathcal{Z}(f^{\varphi})$, $f^{\varphi}(x,y) = f(\varphi^{-1}(x,y))$.

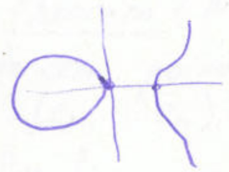
degree 2 (conics):

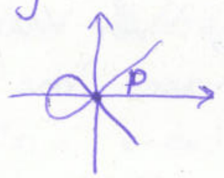
$f(x,y) = ax^2 + bxy + cy^2 + dx + ey + g$.

$a, b, c \neq 0$ (all together)

degree 3 (cubics).

$f = y^2 - x^3 + x$ $\qquad$ $g = y^2 - x^3 - x^2$ $\qquad$ $h = y^2 - x^3$







$\frac{\partial g}{\partial x}(p) = 0 = \frac{\partial g}{\partial y}(p)$.

Def: a point $p \in f$ is a singular point, if $\frac{\partial f}{\partial x}(p) = 0 =$
$= \frac{\partial f}{\partial y}(p)$ (Особая)

Intersection of 2 plane curves $(f, g)$.

$f \cap g = ?$

Special case: $g = \ell$ - line.

1) Chose a linear parametrisation for $\ell$.

$$\ell = \begin{cases} x(t) = \alpha t + \gamma \\ y(t) = \beta t + \delta \end{cases}$$

2) Substitute this parametrisation in $f(x,y) = 0$.

$h(t) = f(x(t), y(t))$

3) Find the roots of $h$.

$p = (x(t_0), y(t_0)) \in \ell \cap f$.

$\Updownarrow$

$t_0$ is a root of $h$.

- $h \equiv 0 \iff \ell$ is an irreducible component of $f$.
- $h \not\equiv 0 \to \deg(h) \leq \deg f$.

$\to \#(f \cap \ell) = \#(\text{roots of } h) \leq \deg h$.

("=", if we count roots with multiplicity).

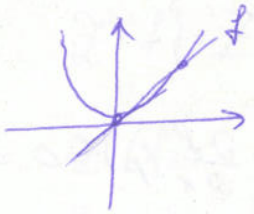
$\#(f\cap \ell)\le d.$

$\#(f\cap \ell)<d$ if
- $h$ has multiple roots
- $\deg h < \deg f = d.$

Assume that $\ell$ is not an irreducible component.

!!! There is a geometris explanation for these !!

<u>Example:</u> $f = y - x^2.$



$\ell$ line through $p=(0;0)$.

$\ell = ax + by$

1) $\begin{cases} x(t) = -bt \\ y(t) = at \end{cases}$

2) $h(t) = at - b^2t^2 = t(a - b^2 t).$

3) Roots of $h$: $\begin{cases} t = 0 \\ t = \frac{a}{b^2}, \; b \neq 0. \end{cases}$

if $a=0$, $h(t) = -b^2t^2$. ← the line is tangent to $f$.

if $b=0$, $h(t) = at$.

<u>Def:</u> The multiplicity of intersection of $f$ and $\ell$ at $p = (x(t_0), y(t_0))$ is $(f\cap\ell)_p$ – multiplicity of $t_0$ at a root of $h(t)$.