①

PLANE ALGEBRAIC CURVES

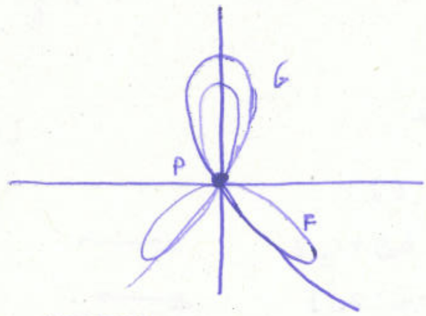
Exercise

$F = (x^2+y^2)^2 - z(y^3 - 3x^2y)$

$G = y^3 - z(y^2 - 3x^2)$

$P = [0:0:1]$

$(F \cap G)_P$



PROPERTIES OF THE MULTIPLICITY OF INTERSECTION

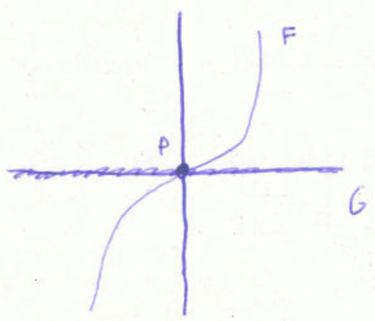
1) invariance under projective equivalence
2) $(F \cap G)_P = 0 \iff P \notin F \cap G$
3) $(x \cap y)_P = 1$
4) $(F \cap GH)_P = (F \cap G)_P + (F \cap H)_P$
5) $(F \cap G + AF)_P = (F \cap G)_P$

Example

$F = yz^2 - x^3$

$G = y$

P is an inflexion



$$(F \cap G)_P \underset{\substack{\uparrow \\ \text{prop 5.}}}{=} (F - z^2G \cap G)_P = (-x^3 \cap G)_P \underset{\substack{\uparrow \\ \text{prop 1, 4}}}{=} 3(x \cap y)_P = 3$$

Exercise: F, G projective plane curves, $p \in F \cap G$ nonsingular $T_pF \neq T_pG$
$\Rightarrow (F \cap G)_p = 1$

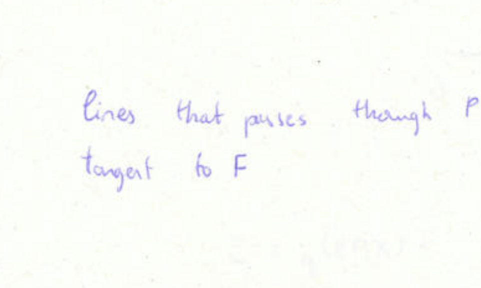
PROBLEM: F irreducible projective plane curve, $P \notin F$
How many lines are tangent to F and pass through P?

$\mathbb{P}^2$		$(\mathbb{P}^2)^\vee$
$[x:y:z]$		$[a:b:c]$
line $\ell = a_0x + b_0y + c_0z$	$\longleftrightarrow$	$Q_\ell = \ell^\vee = [a_0 : b_0 : c_0]$ point
$P = [x_0 : y_0 : z_0]$	$\longleftrightarrow$	$P^\vee = x_0a + y_0b + z_0c$
F	$\longleftrightarrow$	$F^\vee$
$[x:y:z]$	$\longmapsto$	$\left[\frac{\partial F}{\partial x} : \frac{\partial F}{\partial y} : \frac{\partial F}{\partial z}\right]$

$$(F^\vee)^\vee = F$$

The Gauss map:

$$g : F \dashrightarrow (\mathbb{P}^2)^\vee$$
$$p \longmapsto (T_pF)^\vee$$
(p non-singular)



lines that passes through P and tangent to F

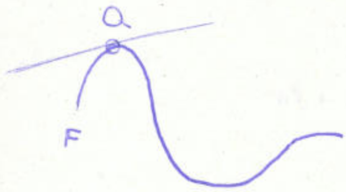
PROBLEM: It's equivalent to: What is the degree of $F^\vee$?

Example

F quadric

$$F = (x\ y\ z)\ A \begin{pmatrix} x \\ y \\ z \end{pmatrix} \qquad A \text{ symmetric and invertible}$$

$$F^\vee = (x\ y\ z)\ A^{-1} \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$



$\cdot P$ $\qquad P = [x_0 : y_0 : z_0]$

$$T_p F = \frac{\partial F}{\partial x}(Q)\cdot x + \frac{\partial F}{\partial y}(Q)\, y + \frac{\partial F}{\partial z}(Q)\cdot z$$

DEFINITION

The polar of F with respect to P is:

$$F^P = x_0 \cdot \frac{\partial F}{\partial x} + y_0 \frac{\partial F}{\partial y} + z_0 \frac{\partial F}{\partial z}$$

curve of degree $d-1$

If $P \in T_Q F$, then $Q \in F^P$

$$F \cap F^P = \left\{ \begin{matrix} \text{singular points} \\ \text{of } F \end{matrix} \right\} \cup \left\{ \begin{matrix} Q \in F \text{ non-} \\ \text{singular st} \\ P \in T_Q F \end{matrix} \right\}$$

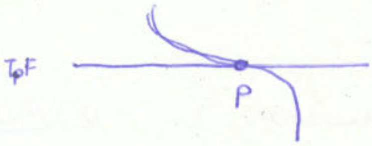
Going back to the original problem. How many lines are tangent to F and pass through P?

- If F is non-singular, then $d^\vee = \deg(F^\vee) = \#(F \cap F^P) = d(d-1)$

Next goal

Compute $(F \cap F^P)_Q$ for any $Q \in F \cap F^P$ (in the non-singular case).

Assumption: P does not belong to any inflection line of F



i.e. $(F \cap T_pF)_p \geq 3$

$Q = [0 : 0 : 1]$

$f = F(x, y, 1) = \underset{\text{multiplicity}}{\overset{\overset{0}{\#}}{f_m}} + f_{m+1} + \text{----} + f_d = \overset{\neq 0}{f_1} + f_2 + \text{---} + f_d$

After a projective equivalence we may assume that $f_1 = y$

$f = y + (ax^2 + bxy + cy^2) + f_3 + \text{-----}$

$P \in T_Q f = y \Rightarrow P = [x_0 : 0 : z_0]$ $\quad (*) \Rightarrow a \neq 0$

$f = y + (ax^2 + bxy + cy^2) + f_3 + \text{-----}$.

$p = (0, 0)$

$(f \cap y)_p$
$\overset{\shortparallel}{T_p f}$

$\boxed{\text{Exercise}}$ is the multiplicity of intersection - ___ ?

$F = z^{d-1} y + z^{d-2} \overset{\overset{0}{\#}}{(ax^2 + bxy + cy^2)} + \text{.....}$

$F^P = x_0 \frac{\partial F}{\partial x} + z_0 \frac{\partial F}{\partial z} = x_0 z^{d-2} (2ax + by) + \text{....}$

$f^P = \underbrace{2ax_0}_{\neq 0} \cdot x + x_0 b \cdot y$ + high other terms

CONCLUSION

f and f^P are nonsingular at Q, with distinct tangent directions
$\Rightarrow (f \cap f^P)_Q = 1$

[If P is general, then $\exists\ d(d-1)$ tangent curves to F passing through P]