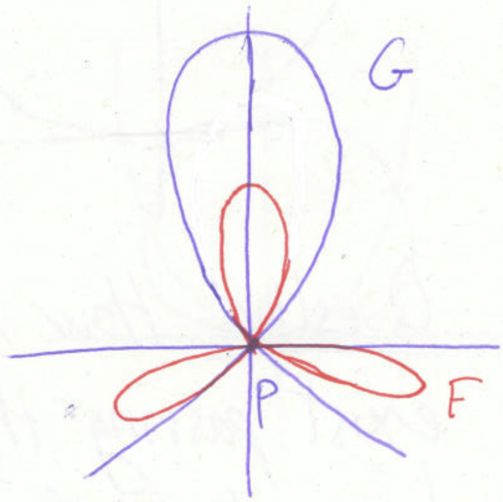
Then the tangent line to F at p is $T_pF = \underbrace{\frac{\partial F}{\partial x}(p)}x + \underbrace{\frac{\partial F}{\partial y}(p)}y + \underbrace{\frac{\partial F}{\partial z}(p)}z$

homogeneus of degree = deg F - 1

---

Exercise: $F = (x^2+y^2)^2 - z(y^3 - 3x^2y)$
$G = y^3 - z(y^2 - 3x^2)$

$p = [0,0,1]$, $(F \cap G)_p$




Properties of the multiplicity of intersection:

(1) invariance under projective equivalence

(2) $(F \cap G)_p = 0 \longleftrightarrow p \notin F \cap G$

(3) $(X \cap Y)_p = 1$

(4) $(F \cap GH)_p = (F \cap G)_p + (F \cap H)_p$

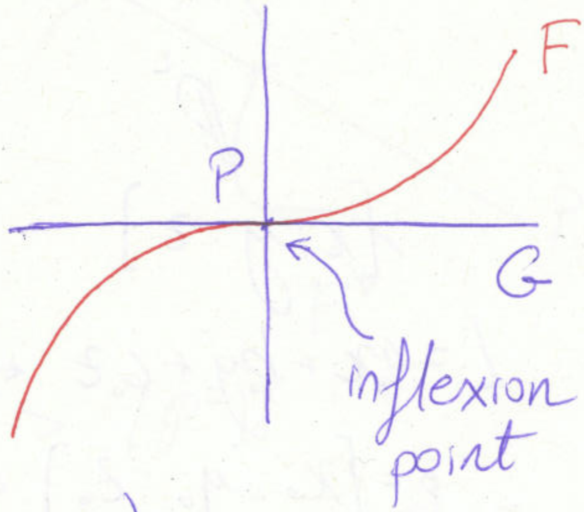
(5) $(F \cap G + AF)_p = (F \cap G)_p$

These properties uniquely determine the definition.

**Example:** $F = yz^2 - x^3$

$G = y$

$(F \cap G)_p$?

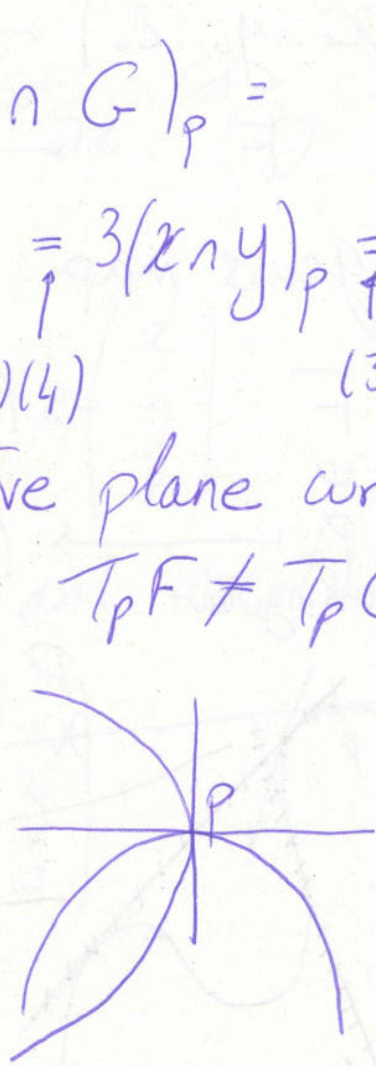



$$(F \cap G)_p \overset{(5)}{=} (F - z^2 G \cap G)_p =$$

$$= (-x^3 \cap G)_p \underset{(1)(4)}{=} 3(x \cap y)_p \underset{(3)}{=} 3$$

**Exercise:** $F, G$ projective plane curves, $p \in F \cap G$ nonsingular $T_pF \neq T_pG$

$\Longrightarrow (F \cap G)_p = 1$



<u>Problem</u>: $F$ irreducible projective plane curve and $p \notin F$. How many lines are tangent to $F$ and pass through $p$.

$$\mathbb{P}^2 \qquad\qquad\qquad (\mathbb{P}^2)^\vee$$

$$[x:y:z] \qquad\qquad\qquad [a:b:c]$$

$$L = a_0x + b_0y + c_0z \longleftrightarrow \ell^\vee = [a_0:b_0:c_0]$$

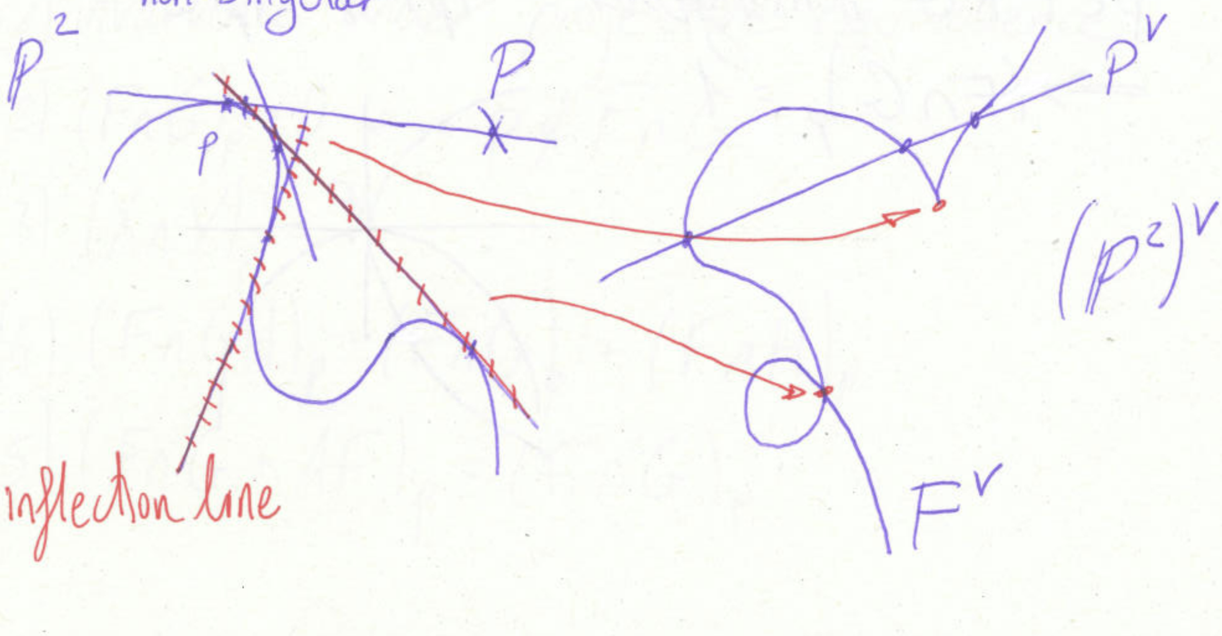
$$p = [x_0:y_0:z_0] \longleftrightarrow p^\vee = x_0a + y_0b + z_0c$$

$$F \longleftrightarrow F^\vee$$

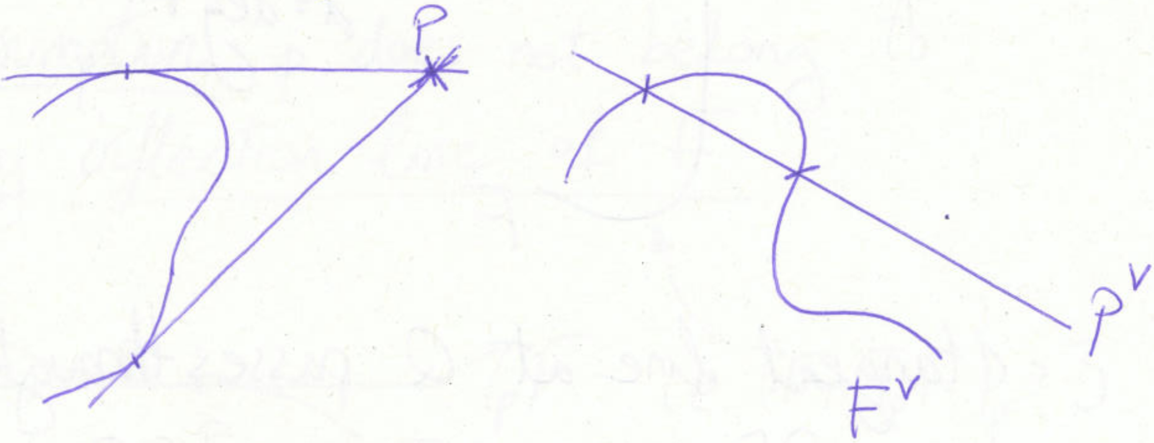
The gauss map:

$$g : F \dashrightarrow (\mathbb{P}^2)^\vee$$
$$p \longmapsto (T_pF)^\vee$$

$p \in F$ non-singular



We have, # lines tangent to $F$ and pass through $p$ = degree of $F^\vee$




$$\mathbb{P}^2 \longrightarrow (\mathbb{P}^2)^\vee$$

$$[x:y:z] \longmapsto \left[\partial F/\partial x : \partial F/\partial y : \partial F/\partial z\right]$$

<u>Example</u>: $F$ quadric

$$F(x,y,z) = (x\ y\ z)\,A\begin{pmatrix}x\\y\\z\end{pmatrix}$$

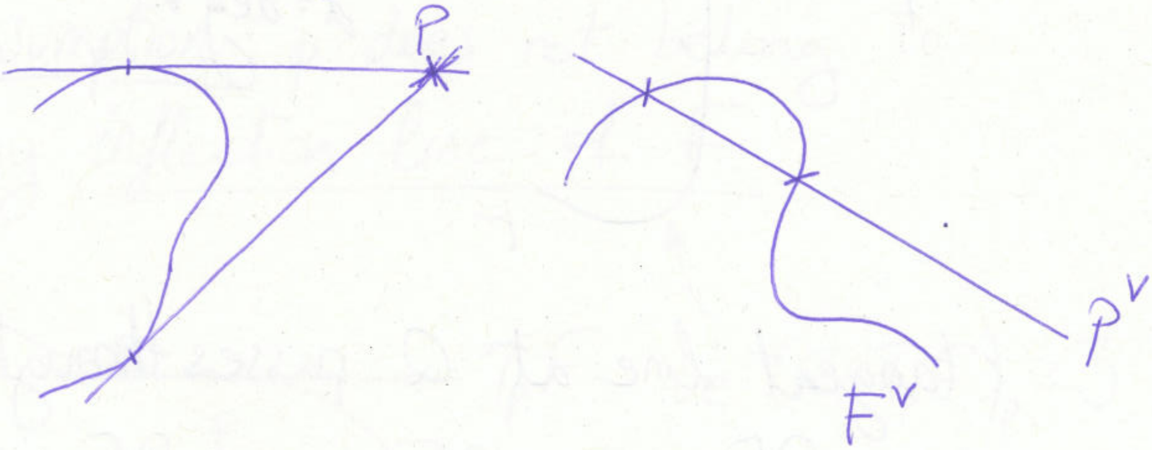
$A$ being symmetric and invertible

$$F^\vee = (x\ y\ z)\,A^{-1}\begin{pmatrix}x\\y\\z\end{pmatrix}$$

$\Longrightarrow$

$$(F^\vee)^\vee = F$$

We have, # lines tangent to F and pass through p ≡ degree of $F^\vee$




$$\mathbb{P}^2 \longrightarrow (\mathbb{P}^2)^\vee$$

$$[x : y : z] \longmapsto \left[ \frac{\partial F}{\partial x} : \frac{\partial F}{\partial y} : \frac{\partial F}{\partial z} \right]$$

<u>Example</u>: F quadric

$$F(x,y,z) = (x\ y\ z)\, A \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

A being symmetric and invertible

$$F^\vee = (x\ y\ z)\, A^{-1} \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

$\Longrightarrow$

$$(F^\vee)^\vee = F$$


$\cdot\, p = [x_0, y_0, z_0]$

$d = \deg F$

¿tangent line at $Q$ passes through $p$?

$$T_Q F = \frac{\partial F}{\partial x}(Q)x + \frac{\partial F}{\partial y}(Q)y + \frac{\partial F}{\partial z}(Q)z$$

Def: The polar of $F$ with respect to $p$ is $F^p = x_0 \frac{\partial F}{\partial x} + y_0 \frac{\partial F}{\partial y} + z_0 \frac{\partial F}{\partial z}$ curve of degree $d-1$

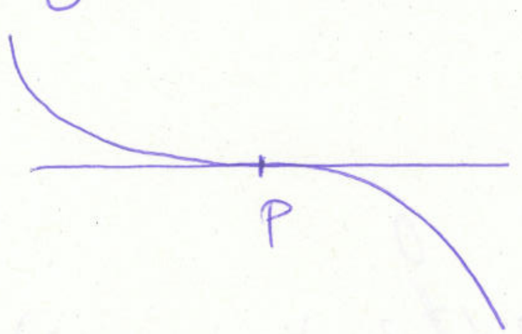
If $p \in T_Q F$, then $Q \in F^p$

$$F \cap F^p = \{\text{singular points of } F\} \cup \{Q \in F \text{ non-singular such that } p \in T_Q F\}$$

If $F$ is non-singular, then $d^\vee = \deg(F^\vee)$ must equal $\#(F \cap F^p) = d(d-1)$ (By Bézout's Theorem)

Next goal: Compute $(F\cap F^p)_Q$ for any $Q\in F\cap F^p$ (in the non-singular case)

Assumption: $p$ does not belong to any inflection line of $F$.



$T_pF$ i.e. $(F\cap T_pF)_p \geq 3$

$Q = [0:0:1]$

$f = F(x,y,1) = f_m + f_{m+1} + \cdots + f_d =$

multiplicity of the curve $f$ at $Q$

non-singular case

$= f_1 + f_2 + \cdots + f_d$

After a projective equivalence, we may assume that $f_1 = y$

$f = y + (ax^2 + 2bxy + cy^2) + f_3 + \cdots$

$p \in T_QF = y \longrightarrow p = [x_0 : 0 : z_0]$

When is $y$ an inflection line?

$f = x + (ax^2 + bxy + cy^2) + f_3 + \cdots$

$p = (0,0)$ Exercise

$\underset{\substack{\| \\ T_p f}}{(f \cap y)_p} = ? \geq 3 \longleftrightarrow a = 0$

Therefore $a \neq 0$

$$F = z^{d-1}y + z^{d-2}(\overset{\substack{0 \\ \#}}{a}x^2 + bxy + cy^2) + \cdots$$

$$F^P = x_0 \frac{\partial F}{\partial x} + z_0 \frac{\partial F}{\partial z} =$$

$$= x_0 z^{d-2}(2ax + by) + \cdots$$

$$f^P = \underset{\substack{\# \\ 0}}{\underline{2ax_0}}x + x_0 by + \text{higher order terms}$$

<u>conclusion</u>: $f$ and $f^P$ are nonsingular at $Q$, with distinct tangent directions

$\Rightarrow (f \cap f^P)_Q = 1$

If $P$ is <u>general</u>, then there exist $d(d-1)$ tangent curves to $F$ passing through $p$.