Def. A projective plane curve is an equivalence class of nonconstant homogeneous polynomials $F \in \mathbb{C}[x,y,z]$ under the equivalence relation that identifies F and $G \leftrightarrow F = \lambda G$ for some $\lambda \in \mathbb{C} \setminus \{0\}$

A projective equivalence of $\mathbb{P}^2$ is is a map $\varphi : \mathbb{P}^2 \longrightarrow \mathbb{P}^2$ induced by $\tilde{\varphi} : \mathbb{C}^3 \longrightarrow \mathbb{C}^3$ invertible linear transformation. F projective curve $\underset{\varphi}{\rightsquigarrow}$

$$\rightsquigarrow F^{\varphi}(x,y,z) = F(\tilde{\varphi}^{-1}(x,y,z))$$

L projective line ($L = ax + by + cz$)

F projective curve

$F \cap L$ 1) linear parametrization for L

$[t,s] \in \mathbb{P}^1$

$x(t,s) = \alpha_1 t + \alpha_2 s$; $y(t,s) = \beta_1 t + \beta_2 s$

$z(t,s) = \gamma_1 t + \gamma_2 s$

2) $H(t,s) = F(x(t,s), y(t,s), z(t,s))$

$H \equiv 0 \longleftrightarrow L$ is an irreducible component of F

If not

H is homogeneous polynomial of degree $=$
$=$ degree F

$H(s,t) = (a_1 t + b_1 s)^{m_1} \cdots (a_k t + b_k s)^{m_k}$

$\sum m_i = \deg F$

Roots of H: $[-b_i, a_i] \in \mathbb{P}^1$

$P_i = [x(-b_i, a_i), y(-b_i, a_i), z(-b_i, a_i)]$
$\cap$
$F \cap L$ point of multiplicity m_i

What about $F \cap G$ for F and G arbitrary projective plane curves?

<u>Bézout's Theorem</u>: Let F and G be projective curves without common components. Then $F \cap G$ consists of exactly $(\deg F)(\deg G)$ points, counted with "multiplicity"

Elimination theory: the resultant

Let A be a ring, (e.g. $A = \mathbb{C}[x,z]$)

$f, g \in A[y] = \mathbb{C}[x,z][y]$

$$f = a_d y^d + a_{d-1} y^{d-1} + \dots + a_0, \quad a_d \neq 0$$

$$g = b_e y^e + b_{e-1} y^{e-1} + \dots + b_0, \quad b_e \neq 0$$

$$\underset{\underset{A}{\Uparrow}}{R_{f,g}} = \det \left[\begin{array}{ll} \left|\begin{matrix} a_d & a_{d-1} & \cdots & a_0 & 0 & \cdots & 0 \\ 0 & a_d & \cdots & a_1 & a_0 & \cdots & 0 \\ \vdots & & \ddots & & & \ddots & \\ 0 & \cdots & & a_d & \cdots & & a_0 \end{matrix}\right\} & e \text{ rows} \\ \left|\begin{matrix} b_e & b_{e-1} & \cdots & b_0 & 0 & \cdots & 0 \\ 0 & b_e & \cdots & b_1 & b_0 & \cdots & 0 \\ & & \ddots & & & & \\ 0 & \cdots & & b_e & \cdots & & b_0 \end{matrix}\right\} & d \text{ rows} \end{array}\right]$$

$M_{f,g}$ $(d+e)\times(d+e)$ matrix

__Proposition__: $R_{f,g} = 0 \longleftrightarrow f$ and g have a common factor (nonconstant)

__Proof__: f and g have a common component $(f = h\tilde{f},\ g = h\tilde{g}) \longleftrightarrow \exists p, q \in A[y]$ such that $\deg p \leq e-1$, $\deg q \leq d-1$ and $pf = qg$

$$p(y) = u_0 y^{e-1} + \dots + u_{e-1}$$

$$q(y) = v_0 y^{d-1} + \dots + v_{d-1}$$

$$\neq (0,\dots,0)$$

$\longleftrightarrow$ $\exists\, u_0,\dots,u_{e-1}, v_0,\dots,v_{d-1}$ such that for $i=0,\dots,d+e-1$

$$\sum_{j=0}^{e-1} v_j a_{d-i-j} = \sum_{=0}^{d-1} u_t b_{e-i-t}$$

The matrix of this linear system is precisely $M_{f,g}$.

$\longleftrightarrow \det M_{f,g} = R_{f,g} = 0$ $\boxtimes$

Back to the intersection of 2 plane curves F and G, $F, G \in \mathbb{C}[x,z][y]$

$$\underset{\text{degree } d}{\overset{\uparrow}{F}} \quad \underset{\deg e}{\overset{\uparrow}{G}}$$

<u>Assumption</u>: $[0,1,0] \notin F, G$

($\longleftrightarrow$ viewed in $\mathbb{C}[x,z][y]$ $\deg F = d$, $\deg G = e$)

$R_{F,G}(x,z) \in \mathbb{C}[x,z]$

If F and G have no common (factor) component, then $R_{F,G}(x,y) \not\equiv 0$ by the previous proposition.

In this case we have the following prop.

Proposition: $R_{F,G}(x,z)$ is a homogeneus polynomial polynomial of degree $d \cdot e$

Proof:

$R_{F,G}(x,z)$ is homogeneus of degree $d \cdot e$
$\longleftrightarrow R_{F,G}(\lambda x, \lambda z) = \lambda^{de} R_{F,G}(x,z)$
(substitute in matrix) $\boxtimes$

Exercise:

$$R_{F,G}(x,z) = \prod_{i=1}^{k} (a_i x + b_i z)^{m_i}$$

$$\sum m_i = d \cdot e$$

$[x_0 : z_0]$ is a root of $R_{F,G}(x,z) \longleftrightarrow$ $F(x_0, y, z_0)$ and $G(x_0, y, z_0)$ have a common factor, of the form $(y - y_0) \longleftrightarrow \exists y_0$ such that $[x_0, y_0, z_0] \in F \cap G$.

F, G projective plane curves of degrees d and e, without common components

Assumption: $[0,1,0] \notin F, G$

$R_{F,G}(x,z)$ homogeneus of degree $d \cdot e$

$R_{F,G}(x_0,z_0)=0 \longleftrightarrow \exists y_0$ such that

$[x_0,y_0,z_0] \in F \cap G$

Example: $F = y^2 + x^2 - 2zx$

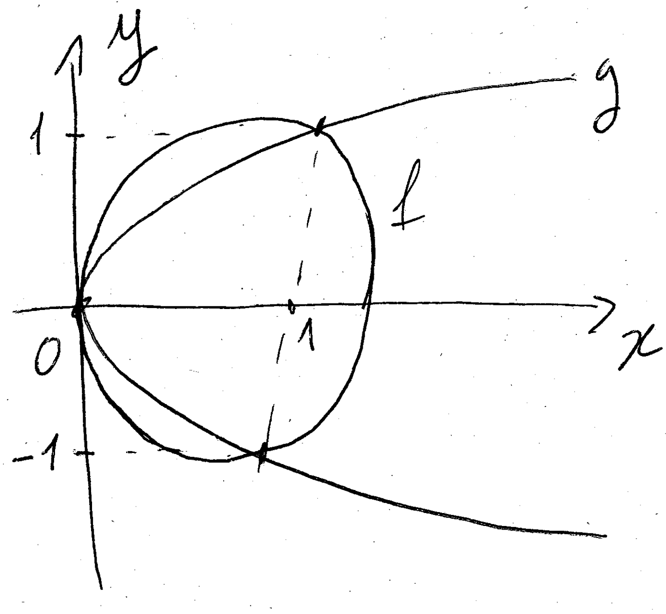
$G = y^2 - xz$

$$R_{F,G} = \det \begin{bmatrix} 1 & 0 & x^2-2zx & 0 \\ 0 & 1 & 0 & x^2-2zx \\ 1 & 0 & -xz & 0 \\ 0 & 1 & 0 & -xz \end{bmatrix} =$$

$$= x^2(x-z)^2 \longleftarrow \begin{matrix}[0,1] \\ [1,1]\end{matrix} \text{ roots}$$

$f = y^2 + x^2 - 2x$

$g = y^2 - x$



View F and G in $\mathbb{C}[y,z][x]$

$$R_{F,G} = \det \begin{bmatrix} 1 & -2z & y^2 \\ -z & y^2 & 0 \\ 0 & -z & y^2 \end{bmatrix} = y^2(y-z)(y+z)$$

roots $[0,1]$ (2)
$[1,1]$ (1)
$[-1,1]$ (1)

Let F and G be projective plane curves without common components and assume (1) $[0,1,0] \notin F, G$

(2) $P \neq Q \in F \cap G \Longrightarrow$
$\longrightarrow [0,1,0] \notin PQ$

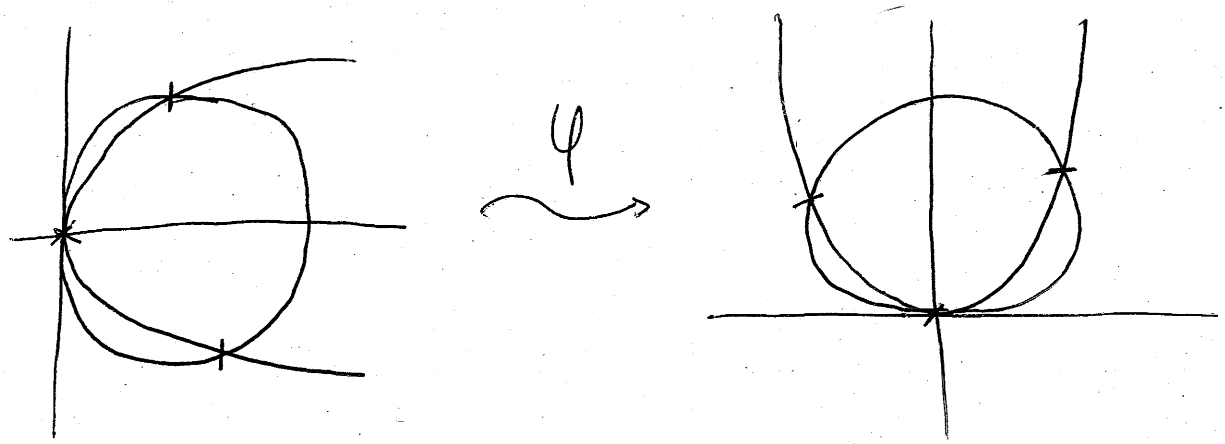
$$R_{F,G}(x,z) = \prod_{i=1}^{k} (a_i x + b_i z)^{m_i}$$

If $[x_0, z_0]$ is a root of $R_{F,G}$ then $\exists$ unique y_0 such that $P = [x_0, y_0, z_0] \in F \cap G$

Def: The multiplicity of intersection of F and G at P is $(F \cap G)_P$ = multiplicity of $[x_0, z_0]$ as a root of $R_{F,G}(x,z)$

Bézout's Theorem: Let F and G be projective plane curves without common components. Then $F \cap G$ consists of exactly $(\deg F)(\deg G)$ points, counted with multiplicity.

If condition not satisfied apply a projective transformation:



Another equivalent definition of multiplicity of intersection

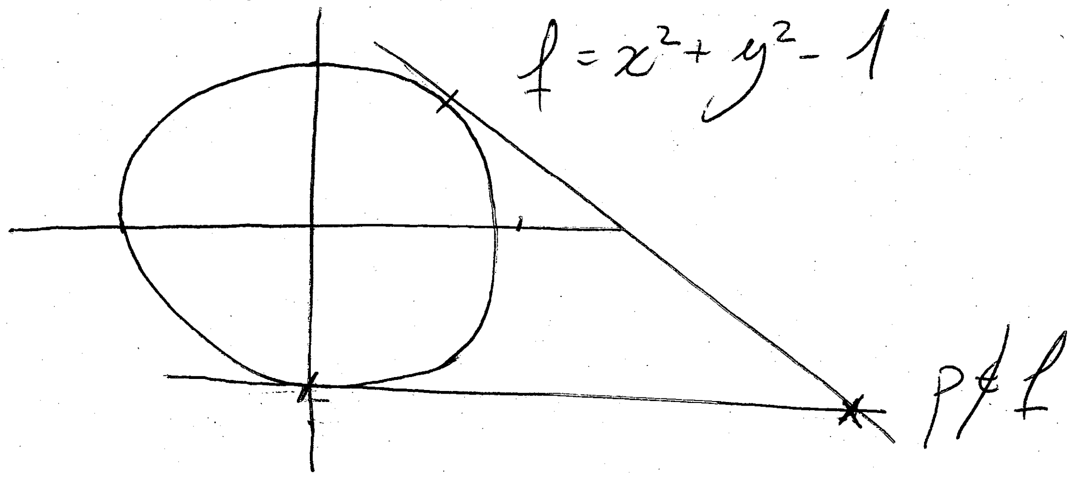
$\mathbb{C}^2$, $p = (0,0) \in f \cap g$

Consider $\underbrace{\mathbb{C}[[x,y]]}$ $\ni f, g$

ring of power series in x and y

$$(f \cap g)_p = \dim_{\mathbb{C}} \left(\mathbb{C}[[x,y]] \big/ (f,g) \right)$$

1st application of Bézout's Theorem
Plücker formulae



<u>Question</u>: How many lines there exist passing through $p \notin f$ that are tangent to f?

Let f be a plane curve and $p = (x_0, y_0) \in f$ non singular point.

F a projective plane curve and $p \in F$ a nonsingular point.

$$\left(\frac{\partial F}{\partial x}(p), \frac{\partial F}{\partial y}(p), \frac{\partial F}{\partial z}(p) \right) \neq (0,0,0)$$

Then the tangent line to F at p

is $T_pF = \underbrace{\frac{\partial F}{\partial x}(p)}x + \underbrace{\frac{\partial F}{\partial y}(p)}y + \underbrace{\frac{\partial F}{\partial z}(p)}z$

homogeneus of degree $= \deg F - 1$