WHAT IS ALGEBRAIC GEOMETRY? PROBLEM SHEET 1

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Exercise 1. Let $f \in \mathbb{C}[x_1, \dots, x_n]$.
If $f(p) = f(p')$ for every $p, p' \in \mathbb{C}^n$ then $f \in \mathbb{C}$.

Exercise 2. Let $T \subset \mathbb{C}[x_1, \dots, x_n]$ and let $I := (T) \subset \mathbb{C}[x_1, \dots, x_n]$ be the ideal generated by T. Prove that $\mathcal{Z}(T) = \mathcal{Z}(I)$

Exercise 3. Let $n \geq 2$. True or false.

(1) For every ideal $I \subsetneq \mathbb{C}[x_1, \dots, x_n]$ the set $\mathcal{Z}(I)$ is non-empty.
(2) For every $f \in \mathbb{C}[x_1, \dots, x_n]$ with f not constant, the set $\mathcal{Z}(f)$ is infinite.

Exercise 4. Let T_1 e T_2 be subsets of $\mathbb{C}[x_1, \dots, x_n]$. Prove that if $T_1 \subset T_2$ then $\mathcal{Z}(T_2) \subset \mathcal{Z}(T_1)$.
Show that the opposite implication fails.

Exercise 5. Let $I \subsetneq \mathbb{C}[x_1, \dots, x_n]$ be a proper ideal generated by homogeneous polpolynomials (a so-called homogeneous ideal). Prove that $(0, \dots, 0) \in \mathcal{Z}(I)$.

Exercise 6. Identify $\mathbb{A}^2$ with $\mathbb{A}^1 \times \mathbb{A}^1$ in the obvious way. Compare the Zariski topology on $\mathbb{A}^2$ with the product topology (on $\mathbb{A}^1 \times \mathbb{A}^1$). Show that they are different and that the Zariski topology is finer than the product topology.

Exercise 7. Let R be a noetherian commutative ring. Prove that every ideal of R is contained in a maximal ideal.

Exercise 8. Let $I \subset \mathbb{C}[x]$ be an ideal. Prove that I is a radical ideal if and only if $I = (f)$ where f is a polynomial whose factorization has the form $f = g_1 \cdot \ldots \cdot g_m$ with $g_i \in \mathbb{C}[x]$ irreducible and distinct up to scalar (i.e. $g_i/g_j \notin \mathbb{C}$ for every $i \neq j$).

Exercise 9. Prove that $\mathcal{I}(X)$ is a radical ideal of $\mathbb{C}[x_1, \dots, x_n]$ for every $X \subset \mathbb{A}^n$.

Exercise 10. Prove that $\sqrt{I} \subset \mathcal{I}(Z(I))$.

Exercise 11. Let I_1 and I_2 be radical ideals in $\mathbb{C}[x]$. Prove that $I_1 + I_2$ is a radical ideal.
Prove that this does not generalize to $\mathbb{C}[x_1, \dots, x_n]$ (not even to $n = 2$).

Exercise 12. Find a maximal ideal in $\mathbb{Q}[x]$ that is not of the form $(x - a)$, with $a \in \mathbb{Q}$.