$$1 = h_1 f_1 + \cdots + h_r f_r + h_{r+1} l$$

where $h_i = h_i(x_1, \dots, x_n, y)$

$$l(\underbrace{x_1, \dots, x_n}_{\underline{x}}, 1/g) = g \cdot 1/g - 1 = 0$$

$$1 = f_1(\underline{x}) h_1\left(\underline{x}, \frac{1}{g(\underline{x})}\right) + \cdots + f_r(\underline{x}) h_r\left(\underline{x}, \frac{1}{g(\underline{x})}\right)$$

$$\Rightarrow 1 = \frac{f_1(\underline{x}) \tilde{h}_1(\underline{x}) + \cdots + f_r(\underline{x}) \tilde{h}_r(\underline{x})}{g^d}$$

$$\Rightarrow g^d = \sum_{i=1}^{r} \tilde{h}_i f_i \Rightarrow g^d \in I \quad \boxtimes$$

LECTURE 4

$X \subset \mathbb{A}^n$ affine subset, then X is a NOETHERIAN topological space, i.e. for all chain

$X \supset X_1 \supset X_2 \supset \cdots \supset X_m \supset \cdots$ X_i closed

then there exists n such that $\forall n' \geq n$ $X_n = X_{n'}$.

This does not happen in $(\mathbb{R}, |\cdot|)$

$[0, 1/n]$: $[0,1] \supset [0, 1/2] \supset \cdots$

Proof: Enough to do that for $X = \mathbb{A}^n$
Use then $\mathbb{C}[x_1, \dots, x_n]$ is noetherian. $\boxtimes$

Def: - X topological space, X is REDUCIBLE if $X = X_1 \cup X_2$: $X_i \subsetneq X$, X_i closed

- If X not reducible we say X is IRREDUCIBLE.

- $Y \subset X$ is a (irreducible) COMPONENT of X if Y is closed, irreducible, and maxima[l] with respect to these properties.

Example: $p \in \mathbb{A}^n$ is irreducible
$f \in R \setminus \mathbb{C}$, $f = g \cdot h$ g, h are distinct irreducible non-constant polynomials
$Z(f) = Z(g) \cup Z(h)$
The two irreducible comp of $Z(f)$)

Lemma: $X \subset \mathbb{A}^n$ affine
X is irreducible $\Leftrightarrow$ $I(X)$ is PRIME

Def: $I \subset R$, I is prime iff $\forall f, g \in R$ $f \cdot g \in I$ at least one of the f and g is in I.

I is prime $\longleftrightarrow$ R/I is a DOMAIN

Proof: If $\mathcal{I}(X)$ is not prime $\Longrightarrow$ $\exists f, g \in R$ such that $f \cdot g \in \mathcal{I}(X)$ but $f, g \notin \mathcal{I}(X)$.

closed

$X_1 = Z(f) \cap X$, $X_2 = Z(g) \cap X$

Claim $X = X_1 \cup X_2$ $\Longrightarrow X$ is reducible

$X_i \subsetneq X$ $(f, g \notin \mathcal{I}(X))$

$\boxed{\Leftarrow}$ Suppose $\mathcal{I}(X)$ is prime

Let $X = X_1 \cup X_2$, X_i closed

Suppose $X_1 \subsetneq X$, we want $X_2 = X$

$\Rightarrow \mathcal{I}(X) \subsetneq \mathcal{I}(X_1)$, $\exists f \in \mathcal{I}(X_1) \setminus \mathcal{I}(X)$

Pick any $g \in \mathcal{I}(X_2)$

Claim $f \cdot g \in \mathcal{I}(X)$ $\left(\begin{array}{l} f \in \mathcal{I}(X_1) \\ g \in \mathcal{I}(X_2) \end{array} \right|$

$$\underset{\substack{\uparrow \\ \mathcal{I}(X) \text{ prime}}}{\Longrightarrow} g \in \mathcal{I}(X) \Rightarrow \mathcal{I}(X_2) \subset \mathcal{I}(X)$$

$$\Rightarrow \mathcal{I}(X_2) = \mathcal{I}(X) \longrightarrow X = X_2 \qquad \boxtimes$$

I prime $\Longrightarrow I = \sqrt{I}$

By H.N.S. the irreducible affine subsets of $\mathbb{A}^n$ are in bijection with prime ideals of R.

<u>Example</u>: $f \in R \setminus \mathbb{C}$

$Z(f) \subset \mathbb{A}^n$ is irreducible $\underset{\Longleftarrow}{\overset{\not\Longrightarrow}{}}$ $f = x_1^d$

f is an irreducible polynomial

$\mathcal{I}(X) = (f)$

$X \subset \mathbb{A}^n$ is irreducible $\longleftrightarrow$ f is an irreducible polynomial.

<u>Proposition</u>: $X \subset \mathbb{A}^n$ affine

Then X has finitely many irreducible components.

<u>Proof</u>: use X is noetherian

X irreducible, $\dim X = ?$

$\dim \mathbb{A}^n = n$

$$\frac{\mathbb{C}[x_1,\dots,x_n]}{\mathcal{I}(X)} =: k[X] \; (= \mathbb{C}[X])$$

$k[X]$ is a domain, a $\mathbb{C}$-algebra

quotient field of $k[X] =: k(X)$ is a field over $\mathbb{C}$.

$\dim X := \operatorname{trdeg}_{\mathbb{C}} k(X)$

$=$

$\mathbb{C} \subset F$ field

example: $F = \mathbb{C}(x,y)$

the transcendence degree of F over $\mathbb{C}$ is the maximum number of algebraically independent (over $\mathbb{C}$) elements in F.

$f_1, \dots, f_t \in F$ are algebraically independent over $\mathbb{C}$ if the only polynomial $P(x_1, \dots, x_t) \in \mathbb{C}[x_1, \dots, x_t]$

such that $P(f_1, \dots, f_t) = 0$ is $P \equiv 0$.

Ex: $\operatorname{trdeg}_{\mathbb{C}} \mathbb{C} = 0$

$\operatorname{trdeg}_{\mathbb{C}} \mathbb{C}(x) = 1$ ($1/x$, x alg. dep)

$\operatorname{trdeg}_{\mathbb{C}} \mathbb{C}(x_1, \dots, x_n) = 1$

- X irreducible, $\dim X = \operatorname{trdeg}_{\mathbb{C}} K(X)$
- X reducible, $X = Y_1 \cup \dots \cup Y_m$

  Y_i irreducible components

  $\dim X = \max\{\dim Y_i\}$